Hindawi
Journal of Function Spaces
Volume 2021, Article ID 4550582, 23 pages
https://doi.org/10.1155/2021/4550582

Research Article

Lump and Interaction Solutions to the (3 + 1)-Dimensional Variable-Coefficient Nonlinear Wave Equation with Multidimensional Binary Bell Polynomials

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Received 4 September 2021; Accepted 11 November 2021; Published 13 December 2021

Academic Editor: Giovanni Di Fratta

In this paper, we study the (3 + 1)-dimensional variable-coefficient nonlinear wave equation which is taken in soliton theory and generated by utilizing the Hirota bilinear technique. We obtain some new exact analytical solutions, containing interaction between a lump-two kink solitons, interaction between two lumps, and interaction between two lumps-soliton, lump-periodic, and lump-three kink solutions for the generalized (3 + 1)-dimensional nonlinear wave equation in liquid with gas bubbles by the Maple symbolic package. Making use of Hirota's bilinear scheme, we obtain its general soliton solutions in terms of bilinear form equation to the considered model which can be obtained by multidimensional binary Bell polynomials. Furthermore, we analyze typical dynamics of the high-order soliton solutions to show the regularity of solutions and also illustrate their behavior graphically.

1. Introduction

It is known that there are a variety of useful and powerful tools to deal with the nonlocal equations, namely, the improved tan $(\phi/2)$-expansion method [1], the homotopy perturbation method [2], Lie symmetry analysis [3], the Bäcklund transformation method [4], the sine-Gordon expansion approach [5], the $(G'/G, 1/G)$, modified (G'/G^2), and $(1/G')$-expansion methods [6], the multiple Exp-function method [7–10], Hirota's bilinear method including the (2 + 1)-dimensional variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation [11], the generalized unstable space time fractional nonlinear Schrödinger equation [12], the inverse Cauchy problems [13], a generalized hyper-elastic rod equation [14], the Kadomtsev-Petviashvili equation [15], the bKP equation [16], the generalized Burgers equation [16], the inverse scattering transformation method [17, 18], and the KP hierarchy reduction method [19]. Combining Hirota's bilinear method with the KP reduction hierarchy method, very recently, Rao et al. [20] discussed the Kadomtsev-Petviashvili-based system and studied the fusion of lumps and line solitons into line solitons, fission of line solitons into lumps and line solitons, a mixture of fission and fusion processes of lumps and line solitons, and the inelastic collision of line rogue waves and line soliton. An improved Hirota bilinear method for the nonlocal complex MKdV equation was constructed in Ref. [21]. Sun et al. [22] investigated a generalized three-component

Hirota-Satsuma coupled KdV equation describing the interactions of two long waves with different dispersion relations by applying Hirota bilinear operator theory. By employing the Hirota bilinear method, Ma [23] constructed the N-soliton solution for three integrable equations in $(2+1)$-dimensions including the $(2+1)$-dimensional KdV equation, the Kadomtsev-Petviashvili equation, and the $(2+1)$-dimensional Hirota-Satsuma-Ito equation and gave the asymptotic analysis of N-soliton solution. Also, by using the bilinear Bell polynomial approach, Cui [24] obtained some new exact solutions for the new extended $(2+1)$-dimensional Boussinesq equation which can be applied to describe the propagation of shallow water waves. And in Ref. [25], Cheng and coauthors obtained the velocity resonance mechanism and the two-, three-, and four-soliton molecules by utilizing the Hirota bilinear method for the combined $(2N+1)$th-order Lax's KdV equation. In [26], the new solitary wave solutions for the $(3+1)$-dimensional extended Jimbo-Miwa equations were investigated. Moreover, Ma obtained lump solutions to a combined fourth-order nonlinear PDE in $(2+1)$-dimensions [27] and interaction solutions to the Hirota-Satsuma-Ito equation in $(2+1)$-dimensions [28].

Consider the $(3+1)$-dimensional variable-coefficient nonlinear wave equation [29–31] which will be investigated below:

$$(u_t+\phi_1(t)uu_x+\phi_2(t)u_{xxx}+\phi_3(t)u_x)_x+\phi_4(t)u_{yy}+\phi_5(t)u_{zz}=0, \tag{1}$$

where $u=u(x,y,z,t)$ is an unknown function and $\phi_j(t)$ $(j=1,\cdots,5)$ are all optional amounts. And u is the wave amplitude, the variable coefficients $\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t)$, and $\phi_5(t)$ denote the bubble-liquid nonlinearity, the bubble-liquid dispersion, the bubble-liquid viscosity, the y-transverse-perturbation, and the z-transverse-perturbation, respectively, and they are all real functions of t. Equation (1) can be transformed to the following:

(i) The well-known constant-coefficient Kadomtsev-Petviashvili equation with $\phi_1(t)=\pm 6, \phi_2(t)=1, \phi_4(t)=3, \phi_3(t)=\phi_5(t)=0$ as

$$(u_t\pm 6uu_x+u_{xxx})_x+3u_{yy}=0 \tag{2}$$

(ii) The cylindrical KdV equation with $\phi_1(t)=\pm 6, \phi_2(t)=1, \phi_3(t)=1/2t, \phi_4(t)=\phi_5(t)=0$ as

$$u_t\pm 6uu_x+u_{xxx}+\frac{1}{2t}u_x=0 \tag{3}$$

For equation (1), some solutions including the multisoliton, Bäcklund transformation, infinite conservation laws, lump solutions, and other soliton wave solutions have been investigated in Refs. [29, 30]. Also, Guo and Chen [31] studied equation (1) and obtained the multisoliton solutions and periodic solutions including X-periodic, Y-periodic, and 2-periodic wave solutions. The solitons, periodic, and travelling waves of a generalized $(3+1)$-dimensional variable-coefficient nonlinear wave equation in liquid with gas bubbles were caught by Deng and Gao [29]. In [32], the first-order lump wave solution and second-order lump wave solution according to the corresponding two-soliton solution and four-soliton solution were presented. The multisoliton solutions and periodic solutions for the $(3+1)$-dimensional variable-coefficient nonlinear wave equation in liquid with gas bubbles were reported by Guo and Chen [31]. Two different types of bright solutions for the generalized $(3+1)$-dimensional nonlinear wave equation by the traveling wave method were obtained by Guo and Chen [33]. The special N wave solutions by applying the linear superposition principle, the resonant multiple wave solutions, and complexiton solutions were investigated for the generalized $(3+1)$-dimensional nonlinear wave in liquid with gas bubbles in [34]. Ma and coauthors probed and analyzed N-soliton solutions and the Hirota conditions in $(1+1)$-dimensions [13] and $(1+2)$-dimensions [35].

During the last years, the various analytical methods were developed to find the exact solutions by powerful scholars for interesting fields of research because of their wide number of applications in the engineering and manufacturing fields, nonlinear models, for example, nonlinear Schrodinger equation [36], the conformable nonlinear differential equation governing wave-propagation in low-pass electrical transmission lines [37], the $(2+1)$-dimensional coupled variant Boussinesq equations [38], the nonlinear directional couplers with metamaterials by including spatial-temporal fractional beta derivative evolution [39], a new $(3+1)$-dimensional Hirota bilinear equation [40], oblique resonant nonlinear waves with dual-power law nonlinearity [41], the coupled Schrödinger-Boussinesq system with the beta derivative [42], and the Hirota-Maccari system [43].

The major aim of this paper is to obtain some novel exact analytical solutions, including interaction between a lump-two kink solitons, interaction between two lumps, and interaction between two lumps-soliton, lump-periodic, and lump-three kink solutions for the $(3+1)$-dimensional variable-coefficient (VC) nonlinear wave equation in liquid with gas bubbles through the method of the bilinear analysis.

The rest of this article is organized as follows. In Section 2, explanations of multidimensional binary Bell polynomials are given. Also, in Section 3, the bilinear form equation to the $(3+1)$-dimensional VC nonlinear wave equation is constructed. In Section 4, we obtain the interaction between a lump-two kink solitons, interaction between two lumps, and interaction between two lumps-soliton, lump-periodic, and lump-three kink solutions along with depicting 3D, density, and 2D graphs for the VC nonlinear wave equation. The conclusion is given in Section 5.

2. Multidimensional Binary Bell Polynomials

Based on Ref. [16, 44, 45], consider $\xi=\xi(x_1,x_2,\cdots,x_n)$ a C^∞ function with multivariables; the polynomials of the following form

$$Y_{n_1x_1,\cdots,n_jx_j}(\xi) \equiv Y_{n_1,\cdots,n_j}\left(\xi_{s_1x_1,\cdots,s_jx_j}\right) = e^{-\xi}\partial_{x_1}^{n_1}\cdots\partial_{x_j}^{n_j}e^{\xi} \quad (4)$$

are called the multidimensional Bell polynomials:

$$\xi_{s_1x_1,\cdots,s_jx_j} = \partial_{x_1}^{s_1}\cdots\partial_{x_j}^{s_j}\xi, \xi_{0x_i} \equiv \xi, s_1 = 0,\cdots,n_1;\cdots;s_j = 0,\cdots,n_j, \quad (5)$$

and we have

$$\begin{gathered} Y_1(\xi) = \xi_x, Y_2(\xi) = \xi_{2x} + \xi_x^2, Y_3(\xi) = \xi_{3x} + 3\xi_x\xi_{2x} + \xi_x^3, \cdots, \xi = \xi(x,t), \\ Y_{x,t}(\xi) = \xi_{x,t} + \xi_x\xi_t, Y_{2x,t}(\xi) = \xi_{2x,t} + \xi_{2x}\xi_t + 2\xi_{x,t}\xi_x + \xi_x^2\xi_t, \cdots. \end{gathered} \quad (6)$$

The multidimensional binary Bell polynomials can be written as

$$\Sigma_{n_1x_1,\cdots,n_jx_j}(\alpha,\beta) = Y_{n_1,\cdots,n_j}(\xi)\Big|_{\xi_{s_1x_1,\cdots,s_jx_j}=\begin{cases}\alpha_{s_1x_1,\cdots,s_jx_j}, & s_1+s_2+\cdots+s_j \text{ is odd,}\\ \beta_{s_1x_1,\cdots,s_jx_j}, & s_1+s_2+\cdots+s_j \text{ is even.}\end{cases}} \quad (7)$$

We have the following conditions as

$$\Sigma_x(\alpha) = \alpha_x, \Sigma_{2x}(\alpha,\beta) = \beta_{2x} + \alpha_x^2, \Sigma_{x,t}(\alpha,\beta) = \beta_{x,t} + \alpha_x\alpha_t, \cdots. \quad (8)$$

Proposition 1. *Suppose* $\alpha = \ln(\Theta/\Delta)$, $\beta = \ln(\Theta\Delta)$, *then the relations between binary Bell polynomials and Hirota D-operator can be written as below:*

$$\Sigma_{n_1x_1,\cdots,n_jx_j}(\alpha,\beta)\Big|_{\alpha=\ln(\Theta/\Delta),\beta=\ln(\Theta\Delta)} = (\Theta\Delta)^{-1}D_{x_1}^{n_1}\cdots D_{x_j}^{n_j}\Theta\Delta, \quad (9)$$

with Hirota operator

$$\prod_{i=1}^{j}D_{x_i}^{n_i}g\cdot\eta = \prod_{i=1}^{j}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i'}\right)^{n_i}\Theta(x_1,\cdots,x_j)\Delta\left(x_1',\cdots,x_j'\right)\Big|_{x_1=x_1',\cdots,x_j=x_j'}. \quad (10)$$

Proposition 2. *Take* $\Xi(\gamma) = \sum_i \delta_i \mathfrak{P}_{s_1x_1,\cdots,s_jx_j} = 0$ *and* $\alpha = \ln(\Theta/\Delta)$, $\beta = \ln(\Theta\Delta)$, *we have*

$$\begin{cases}\sum_i \delta_{1i}Y_{n_1x_1,\cdots,n_jx_j}(\alpha,\beta) = 0,\\ \sum_i \delta_{1i}Y_{s_1x_1,\cdots,s_jx_j}(\alpha,\beta) = 0,\end{cases} \quad (11)$$

which need to satisfy

$$\mathfrak{R}\left(\gamma',\gamma\right) = \mathfrak{R}\left(\gamma'\right) - \mathfrak{R}(\gamma) = \mathfrak{R}(\beta+\alpha) - \mathfrak{R}(\beta-\alpha) = 0. \quad (12)$$

The generalized Bell polynomials $Y_{n_1x_1,\cdots,n_jx_j}(\xi)$ *are as*

$$\begin{aligned}(\Theta\Delta)^{-1}D_{x_1}^{n_1}\cdots D_{x_j}^{n_j}\Theta\Delta &= \Sigma_{n_1x_1,\cdots,n_jx_j}(\alpha,\beta)\Big|_{\alpha=\ln(\Theta/\Delta),\beta=\ln(\Theta\Delta)}\\ &= \Sigma_{n_1x_1,\cdots,n_jx_j}(\alpha,\alpha+\gamma)\Big|_{\alpha=\ln(\Theta/\Delta),\gamma=\ln(\Theta\Delta)}\\ &= \sum_{k_1}^{n_1}\cdots\sum_{k_j}^{n_j}\prod_{i=1}^{j}\binom{n_i}{k_i}\mathfrak{P}_{k_1x_1,\cdots,k_jx_j}(\gamma)Y_{(n_1-k_1)x_1,\cdots,(n_j-k_j)x_j}(\alpha).\end{aligned} \quad (13)$$

The Cole-Hopf transformation will be as

$$\begin{aligned}&Y_{k_1x_1,\cdots,k_jx_j}(\alpha = \ln(\varphi)) = \frac{\varphi_{n_1x_1,\cdots,n_jx_j}}{\varphi},\\ &(\Theta\Delta)^{-1}D_{x_1}^{n_1}\cdots D_{x_j}^{n_j}\Theta\Delta\Big|_{\Delta=\exp(\gamma/2),\Theta/\Delta=\varphi}\\ &\quad = \varphi^{-1}\sum_{k_1}^{n_1}\cdots\sum_{k_j}^{n_j}\prod_{d=1}^{j}\binom{n_d}{k_d}\mathfrak{P}_{k_1x_1,\cdots,k_dx_d}(\gamma)\varphi_{(n_1-k_1)x_1,\cdots,(n_d-k_d)x_d},\end{aligned} \quad (14)$$

with

$$\begin{aligned}Y_t(\alpha) &= \frac{\varphi_t}{\varphi},\\ Y_{2x}(\alpha,\beta) &= \gamma_{2x} + \frac{\varphi_{2x}}{\varphi},\\ Y_{2x,y}(\alpha,\beta) &= \frac{\gamma_{2x}\varphi_y}{\varphi} + \frac{2\gamma_{x,y}\varphi_x}{\varphi} + \frac{\varphi_{2x,y}}{\varphi}.\end{aligned} \quad (15)$$

3. Bilinear Form Equation to the $(3+1)$D VC Nonlinear Wave Equation

To find the linearizing representation, we consider the below:

$$\begin{aligned}u &= c\gamma_{xx} + u_0,\\ \gamma &= \gamma(x,y,z,t),\\ c &= c(t).\end{aligned} \quad (16)$$

Inserting equation (16) into equation (1), one obtains

$$\begin{aligned}\mathfrak{R}(\gamma) = {}&\left(\frac{d}{dt}c(t)\right)\frac{\partial}{\partial x}\gamma(x,y,z,t)\\ &+ c(t)\frac{\partial^2}{\partial x\partial t}\gamma(x,y,z,t) + \phi_1(t)(c(t))^2\left(\frac{\partial}{\partial x}\gamma(x,y,z,t)\right)^2\\ &+ \phi_1(t)(c(t))^2\gamma(x,y,z,t)\frac{\partial^2}{\partial x^2}\gamma(x,y,z,t)\\ &+ \phi_2(t)c(t)\frac{\partial^4}{\partial x^4}\gamma(x,y,z,t) + \phi_3(t)c(t)\frac{\partial^2}{\partial x^2}\gamma(x,y,z,t)\\ &+ \phi_4(t)c(t)\frac{\partial^2}{\partial y^2}\gamma(x,y,z,t) + \phi_5(t)c(t)\frac{\partial^2}{\partial z^2}\gamma(x,y,z,t) = 0,\end{aligned} \quad (17)$$

with

$$c = 12. \tag{18}$$

The new equation $\mathfrak{R}(\gamma)$ is as

$$\mathfrak{R}(\gamma) = \mathfrak{P}_{x,t} + \phi_2(t)\left(\mathfrak{P}_{4x} + 3\mathfrak{P}_{2x}^2\right) + \phi_3(t)\mathfrak{P}_{2x} + \phi_4(t)\mathfrak{P}_{2y} + \phi_5(t)\mathfrak{P}_{2z} = 0. \tag{19}$$

Applying a change of dependent variable

$$\gamma = \ln (g) \Longleftrightarrow u = 12 \ln (g)_{xx}. \tag{20}$$

Theorem 3. *With the following relations*

$$\gamma = \ln (g) \Longleftrightarrow u = 12 \ln (g)_{xx}, \tag{21}$$

into equation (1), the $(3+1)$D VC nonlinear wave equation can be linearized as the following bilinear equation:

$$\mathfrak{R}(g) = (gg_{xt} - g_x g_t) + \phi_2(t)\left(gg_{4x} - 4g_x g_{3x} + 3g_{xx}^2\right) + \phi_3(t)\left(gg_{xx} - g_x^2\right), \tag{22}$$

$\phi_4(t)(gg_{yy} - g_y^2) + \phi_5 \quad (t)(gg_{zz} - g_z^2) = 1/2(D_x D_t + \phi_2(t) D_x^4 + \phi_3(t)D_x^2 + \phi_4(t)D_y^2 + \phi_5(t)D_z^2)g \cdot g = 0,$ *where* $g = g(x, y, z, t)$ *and* $\gamma = \gamma(x, y, z, t)$.

4. Lump, Lump-Kink, and Other Wave Solutions

We would like to derive the general soliton solutions containing interaction between lump-two kink solitons, interaction between two lumps, and interaction between two lumps-soliton, lump-periodic, and lump-three kink solutions.

4.1. Interaction between a Lump-Two Kink Soliton Solutions. In this section, we would like to present the general solutions of the $(3+1)$-dimensional variable-coefficient nonlinear wave equation through utilizing the bilinear method as the below frame:

$$\begin{aligned} g &= \tau_1^2 + \tau_2^2 + \exp(\tau_3) + \exp(\tau_4) + \exp(\tau_3 + \tau_4) + \varepsilon_5(t), \tau_j \\ &= \alpha_j x + \beta_j y + \delta_j z + \varepsilon_j(t), j \\ &= 1, 2, 3, \varepsilon_0 > 0. \end{aligned} \tag{23}$$

The values $\alpha_i, \beta_i, \delta_i, \varepsilon_i(t)$ $(i = 1, 2, 3)$ are real constants to be computed. By substituting (23) into (22), we obtain a system containing 42 nonlinear equations. By solving the nonlinear system, the determined coefficients will be got as the below cases.

Type I

$$\begin{aligned} \varepsilon_1(t) &= \varepsilon_3(t) = \varepsilon_4(t) = \varepsilon_5(t) = 0, \varepsilon_2(t) \\ &= \int -\frac{{\alpha_2}^2\phi_2(t) + {\beta_2}^2\phi_3(t)}{\alpha_2} dt, \alpha_1 = \alpha_3 = \alpha_4 = 0, \alpha_2 = \alpha_2, \\ \beta_1 &= \beta_3 = \beta_4 = 0, \beta_2 = \beta_2, \delta_1 = \delta_1, \delta_2 = \delta_2, \delta_3 = \delta_3, \delta_4 = \delta_4. \end{aligned} \tag{24}$$

The solutions are given as follows:

$$\begin{aligned} u_1 = & \frac{24\alpha_2^2}{{\delta_1}^2 z^2 + \left(\alpha_2 x + \beta_2 y + \delta_2 z + \int -{\alpha_2}^2\phi_2(t) + {\beta_2}^2\phi_3(t)/\alpha_2 dt\right)^2 + e^{\delta_3 z} + e^{\delta_4 z} + e^{\delta_3 z + \delta_4 z}} \\ & - \frac{48\left(\alpha_2 x + \beta_2 y + \delta_2 z + \int -{\alpha_2}^2\phi_2(t) + {\beta_2}^2\phi_3(t)/\alpha_2 dt\right)^2 {\alpha_2}^2}{\left({\delta_1}^2 z^2 + \left(\alpha_2 x + \beta_2 y + \delta_2 z + \int -{\alpha_2}^2\phi_2(t) + {\beta_2}^2\phi_3(t)/\alpha_2 dt\right)^2 + e^{\delta_3 z} + e^{\delta_4 z} + e^{\delta_3 z + \delta_4 z}\right)^2}. \end{aligned} \tag{25}$$

If $\tau_1^2 + \tau_2^2 + \exp(\tau_3) + \exp(\tau_4) + \exp(\tau_3 + \tau_4) + \varepsilon_5(t) \longrightarrow \infty$, the lump solutions $u \longrightarrow 0$ at any t. By selecting the parameters $\delta_1 = 1, \delta_2 = 2, \delta_3 = 1.5, \delta_4 = 3, \alpha_2 = 1, \beta_2 = 3, \phi_2(t) = \cos(t), \phi_3(t) = \sin(t), x = 1, y = 1$, then plots of equation (25) are plotted in Figure 1. And also, by selecting the parameters $\delta_1 = 1, \delta_2 = 2, \delta_3 = 1.5, \delta_4 = 3, \alpha_2 = 1, \beta_2 = 3, \phi_2(t) = 1/4\cos(2t), \phi_3(t) = 1/4\sin(1 + 2t), x = 1, y = 1$, then plots of equation (25) are plotted in Figure 2.

Type II

$$\begin{aligned} \varepsilon_1(t) &= \int -2\frac{\delta_1\phi_4(t)(\beta_1\delta_2 - \beta_2\delta_1)}{\alpha_2\beta_1} dt, \varepsilon_2(t) \\ &= \int -\frac{{\alpha_2}^2{\beta_1}^2\phi_2(t) + {\beta_1}^2{\delta_2}^2\phi_4(t) - {\beta_2}^2{\delta_1}^2\phi_4(t)}{{\beta_1}^2\alpha_2} dt, \varepsilon_4(t) = \varepsilon_5(t) = 0, \\ \varepsilon_3(t) &= \int -2\frac{\beta_3\delta_1\phi_4(t)(\beta_1\delta_2 - \beta_2\delta_1)}{{\beta_1}^2\alpha_2} dt, \alpha_1 \\ &= \alpha_3 = \alpha_4 = 0, \alpha_2 = \alpha_2, \beta_1 = \beta_1, \beta_2 = \beta_2, \beta_3 = \beta_3, \beta_4 = 0, \end{aligned}$$

$$\begin{aligned} \delta_1 &= \delta_1, \delta_2 = \delta_2, \delta_3 = \frac{\beta_3\delta_1}{\beta_1}, \delta_4 \\ &= 0, \phi_1(t) = 0, \phi_3(t) = -\frac{{\delta_1}^2\phi_4(t)}{{\beta_1}^2}. \end{aligned} \tag{26}$$

The solutions are given as follows:

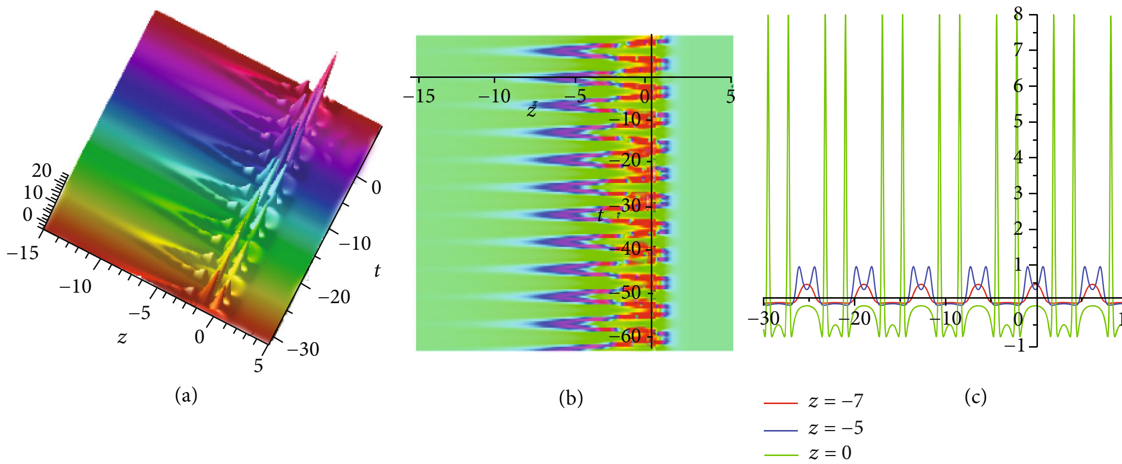


Figure 1: Plots of interaction lump with two solitons (25).

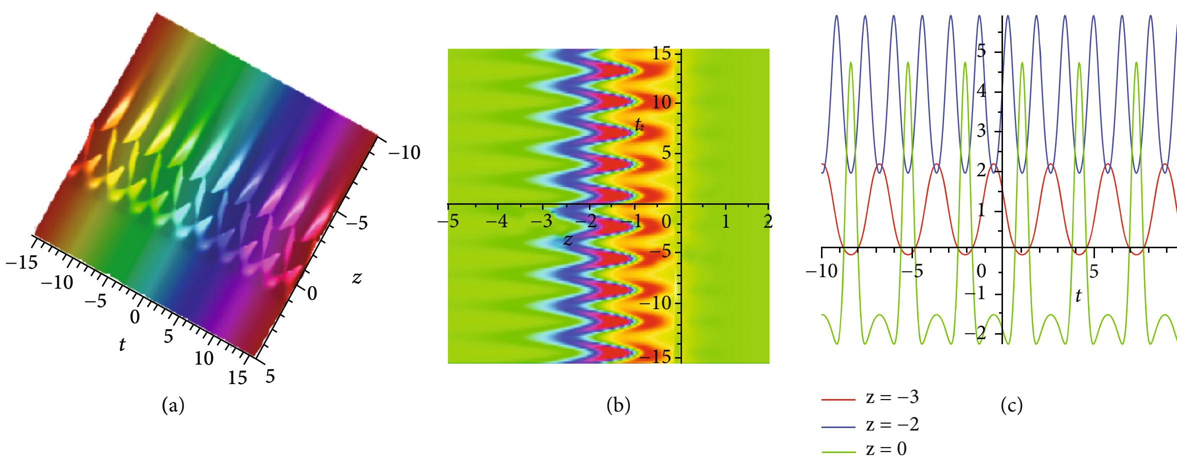


Figure 2: Plots of interaction lump with two solitons (25).

$$u_2 = \frac{24\alpha_2^2}{\tau_1^2 + \tau_2^2 + 2\, e^{\beta_3 y + \beta_3\delta_1 z/\beta_1 + \int -2\beta_3\delta_1\phi_4(t)(\beta_1\delta_2 - \beta_2\delta_1)/\beta_1^2\alpha_2\, dt}} - \frac{48\,\tau_2^2\alpha_2^2}{\left(\tau_1^2 + \tau_2^2 + 2\, e^{\beta_3 y + \beta_3\delta_1 z/\beta_1 + \int -2\beta_3\delta_1\phi_4(t)(\beta_1\delta_2 - \beta_2\delta_1)/\beta_1^2\alpha_2\, dt}\right)^2}, \quad (27)$$

$$\tau_1 = \beta_1 y + \delta_1 z + \int -2\,\frac{\delta_1\phi_4(t)(\beta_1\delta_2 - \beta_2\delta_1)}{\alpha_2\beta_1}\, dt,\ \tau_2 = \alpha_2 x + \beta_2 y + \delta_2 z + \int -\frac{\alpha_2^2\beta_1^2\phi_2(t) + \beta_1^2\delta_2^2\phi_4(t) - \beta_2^2\delta_1^2\phi_4(t)}{\beta_1^2\alpha_2}\, dt. \quad (28)$$

If $\tau_1^2 + \tau_2^2 + \exp(\tau_3) + \exp(\tau_4) + \exp(\tau_3 + \tau_4) + \varepsilon_5(t) \longrightarrow \infty$, the lump solutions $u \longrightarrow 0$ at any t. By selecting the parameters $\delta_1 = 1, \delta_2 = 2, \alpha_2 = 1, \beta_1 = 2, \beta_2 = 3, \beta_3 = 4$, $\phi_2(t) = \cos(t), \phi_4(t) = \sin(t), x = 1, y = 1$, then plots of equation (27) are plotted in Figure 3.

Type III

$$\varepsilon_1(t) = \int 2\,\frac{\delta_1\phi_4(t)(\beta_2\delta_4 - \beta_4\delta_2)}{\alpha_2\beta_4}\, dt,\ \varepsilon_2(t) = \int -\frac{\alpha_2^2\beta_4^2\phi_2(t) - \beta_2^2\delta_4^2\phi_4(t) + \beta_4^2\delta_2^2\phi_4(t)}{\beta_4^2\alpha_2}\, dt,\ \varepsilon_5(t) = 0,$$

$$\varepsilon_3(t) = \int 2\,\frac{\delta_3\phi_4(t)(\beta_2\delta_4 - \beta_4\delta_2)}{\alpha_2\beta_4}\, dt,\ \varepsilon_4(t) = \int 2\,\frac{\delta_4\phi_4(t)(\beta_2\delta_4 - \beta_4\delta_2)}{\alpha_2\beta_4}\, dt,\ \alpha_1 = \alpha_3 = \alpha_4 = 0,\ \alpha_2 = \alpha_2,$$

$$\beta_1 = \frac{\beta_4\delta_1}{\delta_4},\ \beta_2 = \beta_2,\ \beta_3 = \frac{\delta_3\beta_4}{\delta_4},\ \beta_4 = \beta_4,\ \delta_1 = \delta_1,\ \delta_2 = \delta_2,\ \delta_3 = \delta_3,\ \delta_4 = \delta_4,\ \phi_1(t) = 0,\ \phi_3(t) = -\frac{\delta_4^2\phi_4(t)}{\beta_4^2}. \quad (29)$$

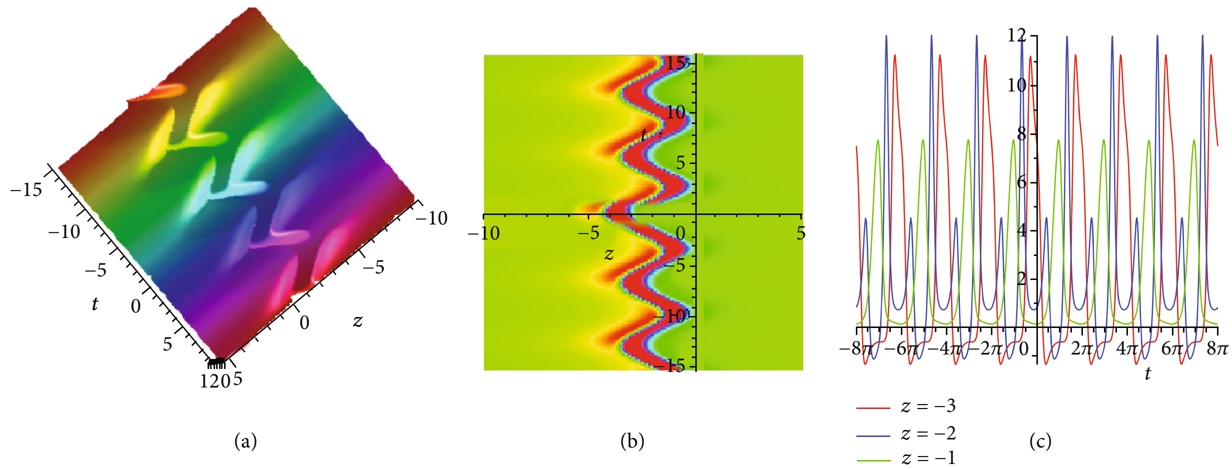


FIGURE 3: Plots of interaction lump with two solitons (27).

The solutions are given as follows:

$$u_3 = 24\frac{\alpha_2^2}{\tau_1^2+\tau_2^2+F_1+F_2+F_3} - 48\frac{\tau_2^2\alpha_2^2}{\left(\tau_1^2+\tau_2^2+F_1+F_2+F_3\right)^2}, \tag{30}$$

$$\tau_1 = \frac{\beta_4\delta_1 y}{\delta_4} + \delta_1 z + \int 2\frac{\delta_1\phi_4(t)(\beta_2\delta_4-\beta_4\delta_2)}{\alpha_2\beta_4}dt, \tau_2 = \alpha_2 x + \beta_2 y + \delta_2 z + \int -\frac{\alpha_2^2\beta_4^2\phi_2(t) - \beta_2^2\delta_4^2\phi_4(t) + \beta_4^2\delta_2^2\phi_4(t)}{\beta_4^2\alpha_2}dt, \tag{31}$$

$$F_1 = e^{\delta_3\beta_4 y/\delta_4+\delta_3 z+\int 2\delta_3\phi_4(t)(\beta_2\delta_4-\beta_4\delta_2)/\alpha_2\beta_4\,dt}, F_2 = e^{\beta_4 y+\delta_4 z+2\delta_4 t\phi_4(t)(\beta_2\delta_4-\beta_4\delta_2)/\alpha_2\beta_4}, \tag{32}$$

$$F_3 = e^{\delta_3\beta_4 y/\delta_4+\delta_3 z+\int 2\delta_3\phi_4(t)(\beta_2\delta_4-\beta_4\delta_2)/\alpha_2\beta_4\,dt+\beta_4 y+\delta_4 z+2\delta_4 t\phi_4(t)(\beta_2\delta_4-\beta_4\delta_2)/\alpha_2\beta_4}. \tag{33}$$

By selecting the parameters $\delta_1 = 1, \delta_2 = 2, \delta_3 = 3, \delta_4 = 4, \alpha_2 = 1, \beta_2 = 3, \beta_4 = 4, \phi_2(t) = \cos(t), \phi_4(t) = \sin(t), x = 1, y = 1$, then plots of equation (30) are plotted in Figure 4.

Type IV

$$\varepsilon_1(t) = -\frac{\varepsilon_2(t)\delta_2}{\delta_1}, \varepsilon_3(t) = \int -\frac{\alpha_3^4\phi_1(t)+\alpha_3^2\phi_2(t)+\beta_3^2\phi_3(t)}{\alpha_3}dt, \varepsilon_4(t) = 0, \varepsilon_5(t) = -2\frac{\varepsilon_2(t)(\delta_1\varepsilon_2(t)-\delta_2\varepsilon_1(t))}{\delta_1},$$

$$\alpha_1 = \alpha_2 = \alpha_4 = 0, \alpha_3 = \alpha_3, \beta_1 = \beta_2 = \beta_4 = 0, \beta_3 = \beta_3, \delta_1 = \delta_1, \delta_2 = \delta_2, \delta_3 = \delta_3, \delta_4 = \delta_4, \phi_4(t) = 0. \tag{34}$$

The solutions are given as follows:

$$u_4 = \frac{12\alpha_3^2F_1 + 12\alpha_3^2F_2}{(\delta_1 z - \varepsilon_2(t)\delta_2/\delta_1)^2 + (\delta_2 z + \varepsilon_2(t))^2 + F_1 + e^{\delta_4 z} + F_2 - 2\varepsilon_2(t)(\delta_1\varepsilon_2(t) - \delta_2\varepsilon_1(t))/\delta_1}, \tag{35}$$

$$\frac{12(\alpha_3F_1+\alpha_3F_2)^2}{\left((\delta_1 z - \varepsilon_2(t)\delta_2/\delta_1)^2 + (\delta_2 z + \varepsilon_2(t))^2 + F_1 + e^{\delta_4 z} + F_2 - 2\varepsilon_2(t)(\delta_1\varepsilon_2(t) - \delta_2\varepsilon_1(t))/\delta_1\right)^2}, \tag{36}$$

$$F_1 = e^{\alpha_3 x+\beta_3 y+\delta_3 z+\int -\alpha_3^4\phi_1(t)+\alpha_3^2\phi_2(t)+\beta_3^2\phi_3(t)/\alpha_3\,dt}, F_2 = e^{\alpha_3 x+\beta_3 y+\delta_3 z+\int -\alpha_3^4\phi_1(t)+\alpha_3^2\phi_2(t)+\beta_3^2\phi_3(t)/\alpha_3\,dt+\delta_4 z}. \tag{37}$$

By selecting the parameters $\delta_1 = 1, \delta_2 = 2, \delta_3 = 3, \delta_4 = 4, \alpha_2 = 1, \beta_2 = 3, \beta_4 = 4, \phi_2(t) = \cos(t), \phi_4(t) = \sin(t), x = 1, y = 1$, then plots of equation (35) are plotted in Figure 5.

Type V

$$\varepsilon_1(t) = -\frac{\varepsilon_2(t)\delta_2}{\delta_1}, \varepsilon_3(t) = \int -\alpha_3\left(\alpha_3^2\phi_1(t)+\phi_2(t)\right)dt, \varepsilon_4(t) = 0, \varepsilon_5(t) = \int -2\frac{(d/dt\varepsilon_2(t))(\delta_1\varepsilon_2(t)-\delta_2\varepsilon_1(t))}{\delta_1}dt,$$

$$\alpha_1 = \alpha_2 = \alpha_4 = 0, \alpha_3 = \alpha_3, \beta_1 = \beta_2 = \beta_4 = 0, \beta_3 = \beta_3, \delta_1 = \delta_1, \delta_2 = \delta_2, \delta_3 = \delta_3, \delta_4 = \delta_4, \phi_3(t) = \phi_4(t) = 0. \tag{38}$$

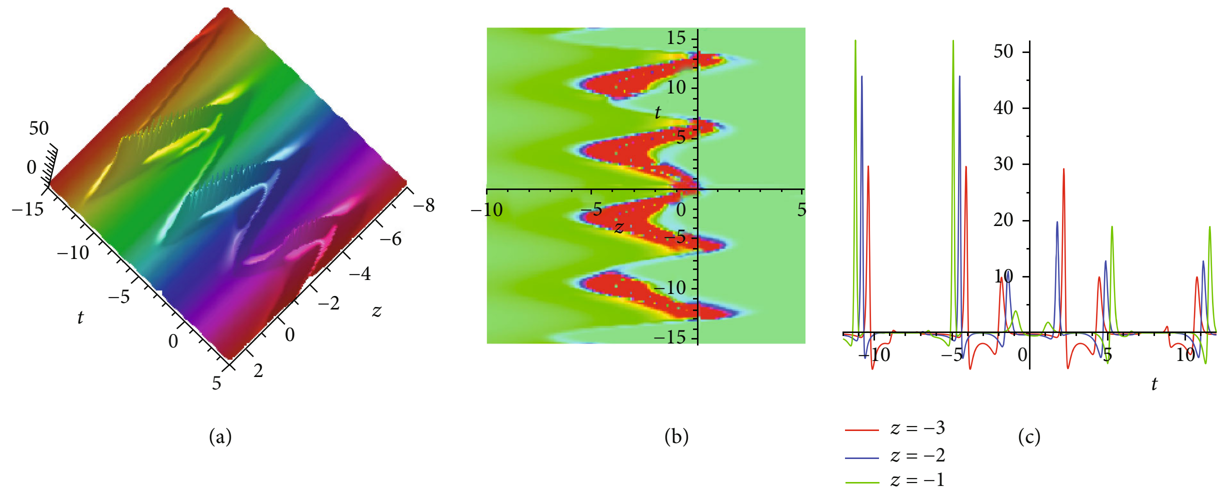


Figure 4: Plots of interaction lump with two solitons (30).

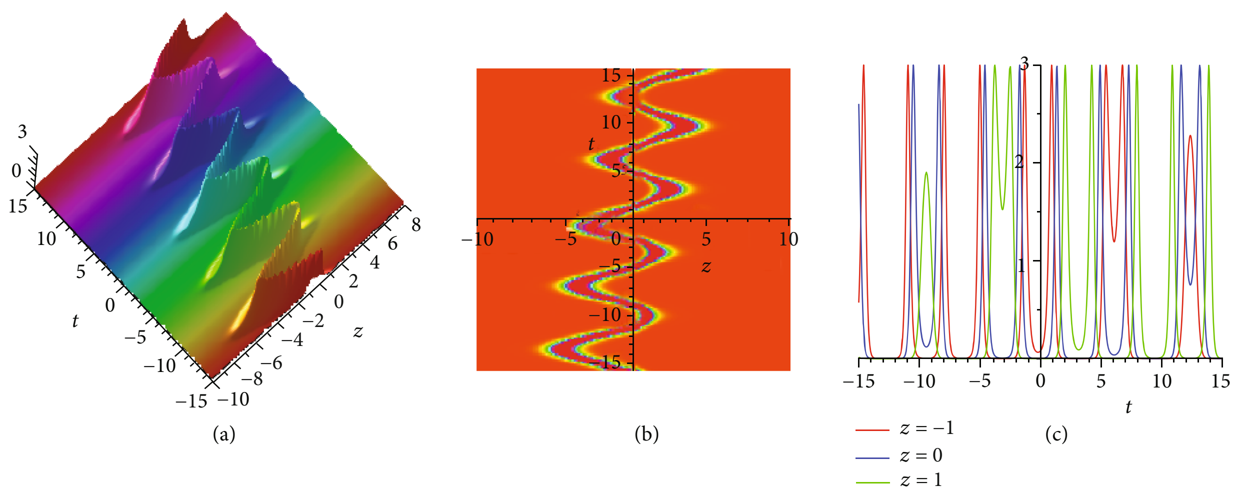


Figure 5: Plots of interaction lump with two solitons (35).

The solutions are given as follows:

$$u_5 = 12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_5),$$
$$\mathbf{F}_5 = (\delta_1 z + \varepsilon_1(t))^2 + (\delta_2 z + \varepsilon_2(t))^2 + e^{\alpha_3 x + \beta_3 y + \delta_3 z + \varepsilon_3(t)} + e^{\beta_4 y + \delta_4 z + \varepsilon_4(t)} + e^{\alpha_3 x + \beta_3 y + \delta_3 z + \varepsilon_3(t) + \beta_4 y + \delta_4 z + \varepsilon_4(t)} + \varepsilon_5(t). \quad (39)$$

Type VI

$$\varepsilon_3(t) = \int -\frac{\alpha_3{}^4\phi_1(t) + \alpha_3{}^2\phi_2(t) + \delta_3{}^2\phi_3(t)}{\alpha_3}\,dt, \varepsilon_2(t) = \varepsilon_4(t) = 0, \varepsilon_5(t) = -(\varepsilon_1(t))^2,$$
$$\alpha_1 = \alpha_2 = \alpha_4 = 0, \alpha_3 = \alpha_3, \beta_1 = 0, \beta_2 = \beta_2, \beta_3 = \beta_3, \beta_4 = \beta_4, \delta_1 = \delta_2 = \delta_4 = 0, \delta_3 = \delta_3, \phi_3(t) = 0. \quad (40)$$

The solutions are given as follows:

$$u_5 = 12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_6), \mathbf{F}_6 = (\beta_2 y + \varepsilon_2(t))^2 + e^{\alpha_3 x + \beta_3 y + \delta_3 z + \varepsilon_3(t)} + e^{\beta_4 y + \varepsilon_4(t)} + e^{\alpha_3 x + \beta_3 y + \delta_3 z + \varepsilon_3(t) + \beta_4 y + \varepsilon_4(t)}. \quad (41)$$

Type VII

$$\varepsilon_3(t) = \int -\alpha_3\left(\alpha_3{}^2\phi_1(t) + \phi_2(t)\right)dt, \varepsilon_1(t) = \varepsilon_2(t) = \varepsilon_4(t) = \varepsilon_5(t) = 0, \alpha_1 = \alpha_2 = \alpha_4 = 0, \alpha_3 = \alpha_3,$$
$$\beta_1 = 0, \beta_2 = \beta_2, \beta_3 = \beta_3, \beta_4 = \beta_4, \delta_1 = \delta_1, \delta_2 = \delta_2, \delta_3 = \delta_3, \delta_4 = \delta_4, \phi_3(t) = \phi_4(t) = 0. \quad (42)$$

The solutions are given as follows:

$$u_7 = 12\frac{\partial^2}{\partial x^2}\ \ln(\mathbf{F}_7),$$
$$\mathbf{F}_7 = z^2{\delta_1}^2 + (\beta_2 y + \delta_2 z)^2 + e^{\alpha_3 x+\beta_3 y+\delta_3 z+\int -\alpha_3\left({\alpha_3}^2\phi_1(t)+\phi_2(t)\right)dt} + e^{\beta_4 y+\delta_4 z} + e^{\alpha_3 x+\beta_3 y+\delta_3 z+\int -\alpha_3\left({\alpha_3}^2\phi_1(t)+\phi_2(t)\right)dt+\beta_4 y+\delta_4 z}. \quad (43)$$

Type VIII

$$\varepsilon_2(t) = \int -2\frac{\delta_2\phi_4(t)(\beta_2\delta_3-\beta_3\delta_2)}{\alpha_3\beta_2}dt, \varepsilon_3(t) = \int -\frac{{\alpha_3}^4{\beta_2}^2\phi_1(t)+{\alpha_3}^2{\beta_2}^2\phi_2(t)+{\beta_2}^2{\delta_3}^2\phi_4(t)-{\beta_3}^2{\delta_2}^2\phi_4(t)}{{\beta_2}^2\alpha_3}dt,$$
$$\varepsilon_4(t) = \int -2\,\frac{(\beta_2\delta_3-\beta_3\delta_2)\delta_4\phi_4(t)}{\alpha_3\beta_2}dt, \varepsilon_5(t) = -\varepsilon_1^2(t), \alpha_1=\alpha_2=\alpha_4=0, \alpha_3=\alpha_3,$$
$$\beta_1=0, \beta_2=\beta_2, \beta_3=\beta_3, \beta_4=\frac{\beta_2\delta_4}{\delta_2}, \delta_1=\delta_1, \delta_2=\delta_2, \delta_3=\delta_3, \delta_4=\delta_4,\ \phi_3(t) = -\frac{{\delta_2}^2\phi_4(t)}{{\beta_2}^2}. \quad (44)$$

The solutions are given as follows:

$$u_8 = 12\frac{\partial^2}{\partial x^2}\ \ln(\mathbf{F}_8),$$
$$\mathbf{F}_8 = (\varepsilon_1(t))^2 + (\beta_2 y+\delta_2 z+\varepsilon_2(t))^2 + e^{\alpha_3 x+\beta_3 y+\delta_3 z+\varepsilon_3(t)} + e^{\beta_2\delta_4 y/\delta_2+\delta_4 z+\varepsilon_4(t)} + e^{\alpha_3 x+\beta_3 y+\delta_3 z+\varepsilon_3(t)+\beta_2\delta_4 y/\delta_2+\delta_4 z+\varepsilon_4(t)} + \varepsilon_5(t). \quad (45)$$

Type IX

$$\varepsilon_1(t) = -\frac{\varepsilon_2(t)\beta_2}{\beta_1}, \varepsilon_3(t) = \int -\alpha_3\left({\alpha_3}^2\phi_1(t)+\phi_2(t)\right)dt, \varepsilon_4(t) = 0, \varepsilon_5(t) = \int -2\frac{(d/dt\varepsilon_2(t))(\beta_1\varepsilon_2(t)-\beta_2\varepsilon_1(t))}{\beta_1}dt,$$
$$\alpha_1=\alpha_2=\alpha_4=0, \alpha_3=\alpha_3, \beta_1=\beta_1, \beta_2=\beta_2, \beta_3=\beta_3, \beta_4=\beta_4,$$
$$\delta_1=\delta_1, \delta_2=\frac{\beta_2\delta_1}{\beta_1}, \delta_3=\delta_3, \delta_4=\delta_4, \phi_3(t)=\phi_4(t)=0. \quad (46)$$

The solutions are given as follows:

$$u_9 = 12\frac{\partial^2}{\partial x^2}\ \ln(\mathbf{F}_9), \mathbf{F}_9 = (\beta_1 y+\delta_1 z+\varepsilon_1(t))^2 + \left(\beta_2 y+\frac{\beta_2\delta_1 z}{\beta_1}+\varepsilon_2(t)\right)^2 + e^{\alpha_3 x+\beta_3 y+\delta_3 z+\varepsilon_3(t)} + e^{\beta_4 y+\delta_4 z+\varepsilon_4(t)} + e^{\alpha_3 x+\beta_3 y+\delta_3 z+\varepsilon_3(t)+\beta_4 y+\delta_4 z+\varepsilon_4(t)} + \varepsilon_5(t). \quad (47)$$

Type X

$$\varepsilon_1(t) = \int -\frac{2\,{\beta_1}^3\delta_1\delta_3\phi_4(t) - 2\,{\beta_1}^2\beta_3{\delta_1}^2\phi_4(t) + 2\,\beta_1{\beta_2}^2\delta_1\delta_3\phi_4(t) - 2\,{\beta_2}^2\beta_3{\delta_1}^2\phi_4(t) + (d/dt\varepsilon_2(t))\alpha_3{\beta_1}^2\beta_2}{\alpha_3{\beta_1}^3}dt,$$
$$\varepsilon_3(t) = \int -\frac{{\alpha_3}^4{\beta_1}^2\phi_1(t)+{\alpha_3}^2{\beta_1}^2\phi_2(t)+{\beta_1}^2{\delta_3}^2\phi_4(t)-{\beta_3}^2{\delta_1}^2\phi_4(t)}{\alpha_3{\beta_1}^2}dt, \varepsilon_4(t) = \int -2\,\frac{(\beta_1\delta_3-\beta_3\delta_1)\delta_4\phi_4(t)}{\alpha_3\beta_1}dt,$$
$$\varepsilon_5(t) = \int -2\,\frac{\left(\varepsilon_2(t){\beta_1}^4\alpha_3+\int S_1+(d/dt\varepsilon_2(t))\alpha_3{\beta_1}^2\beta_2\,dt\beta_2\right)\left(S_2+(d/dt\varepsilon_2(t))\alpha_3{\beta_1}^2\right)}{{\alpha_3}^2{\beta_1}^6}dt, \quad (48)$$
$$S_1 = 2\,\delta_1\phi_4(t)\left({\beta_1}^2+{\beta_2}^2\right)(\beta_1\delta_3-\beta_3\delta_1), S_2 = 2\,\beta_2\delta_1\phi_4(t)(\beta_1\delta_3-\beta_3\delta_1),$$
$$\alpha_1=\alpha_2=\alpha_4=0, \beta_4=\frac{\beta_1\delta_4}{\delta_1}, \delta_1=\delta_1, \delta_2=\frac{\beta_2\delta_1}{\beta_1}, \delta_3=\delta_3, \delta_4=\delta_4, \phi_3(t) = -\frac{{\delta_1}^2\phi_4(t)}{{\beta_1}^2}.$$

The solutions are given as follows:

$$u_{10} = 12\frac{\partial^2}{\partial x^2}\ \ln(\mathbf{F}_{10}), \mathbf{F}_{10} = (\beta_1 y+\delta_1 z+\varepsilon_1(t))^2 + \left(\beta_2 y+\frac{\beta_2\delta_1 z}{\beta_1}+\varepsilon_2(t)\right)^2 + e^{\alpha_3 x+\beta_3 y+\delta_3 z+\varepsilon_3(t)} + e^{\beta_1\delta_4 y/\delta_1+\delta_4 z+\varepsilon_4(t)} + e^{\alpha_3 x+\beta_3 y+\delta_3 z+\varepsilon_3(t)+\beta_1\delta_4 y/\delta_1+\delta_4 z+\varepsilon_4(t)} + \varepsilon_5(t). \quad (49)$$

By selecting the parameters $\delta_1 = 1, \delta_3 = 3, \delta_4 = 4, \alpha_3 = 1, \beta_1 = 4, \beta_2 = 2, \beta_3 = 3, \phi_1(t) = \sin(2t), \phi_2(t) = \cos(2t), \phi_4(t) = \cos(3t), \varepsilon_2(t) = \exp(t), x = 1, y = 1$, then plots of equation (49) are plotted in Figure 6. And also, by choosing the parameters $\delta_1 = 1, \delta_3 = 3, \delta_4 = 4, \alpha_3 = 1, \beta_1 = 4, \beta_2 = 2, \beta_3 = 3, \phi_1(t) = \exp(t)\sin(t), \phi_2(t) = \exp(t)\cos(2t), \phi_4(t) = \exp(t)\cos(3t), \varepsilon_2(t) = \exp(2t), x = 1, y = 1$, then plots of equation (49) are plotted in Figure 7. Moreover, by choosing the parameters $\delta_1 = 1, \delta_3 = 3, \delta_4 = 4, \alpha_3 = 1, \beta_1 = 4, \beta_2 = 2, \beta_3 = 3, \phi_1(t) = 1/1+\sin(t), \phi_2(t) = \cos(t)/(2+\sin(t)), \phi_4(t) = \cos(t), \varepsilon_2(t) = \sin^2(t)+\tan(t), x = 1, y = 1$, then plots of equation (49) are plotted in Figure 8.

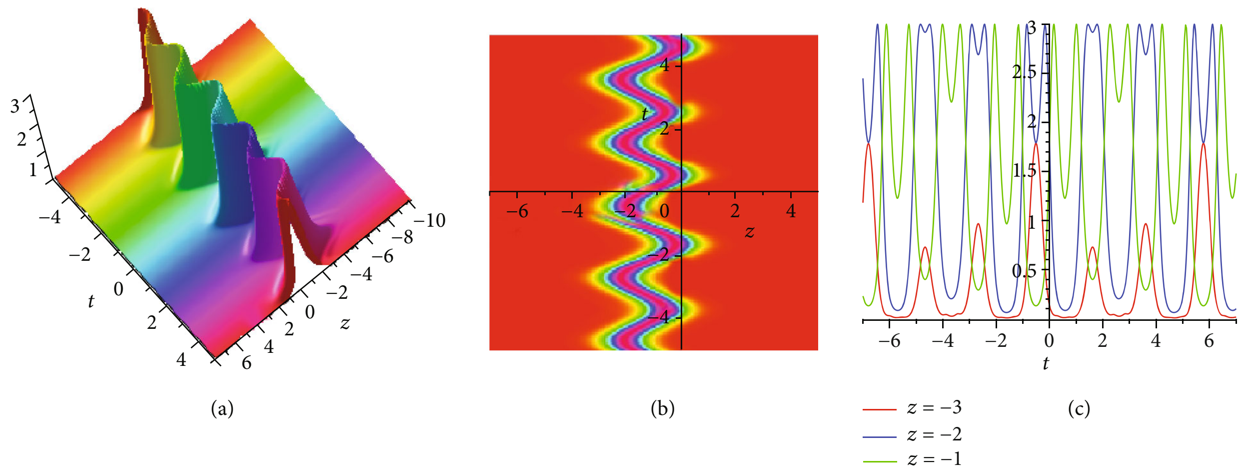


Figure 6: Plots of interaction lump with two solitons (49).

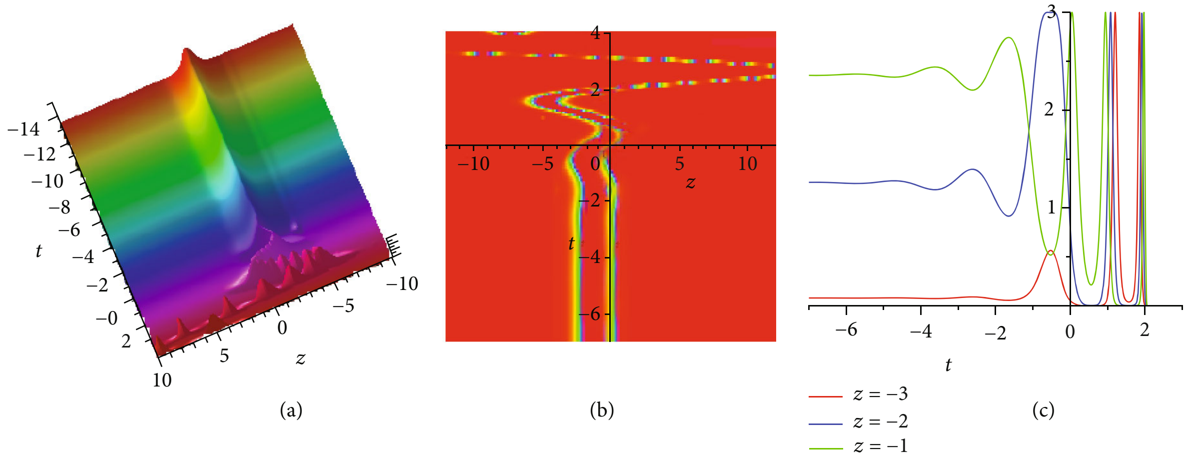


Figure 7: Plots of interaction lump with two solitons (49).

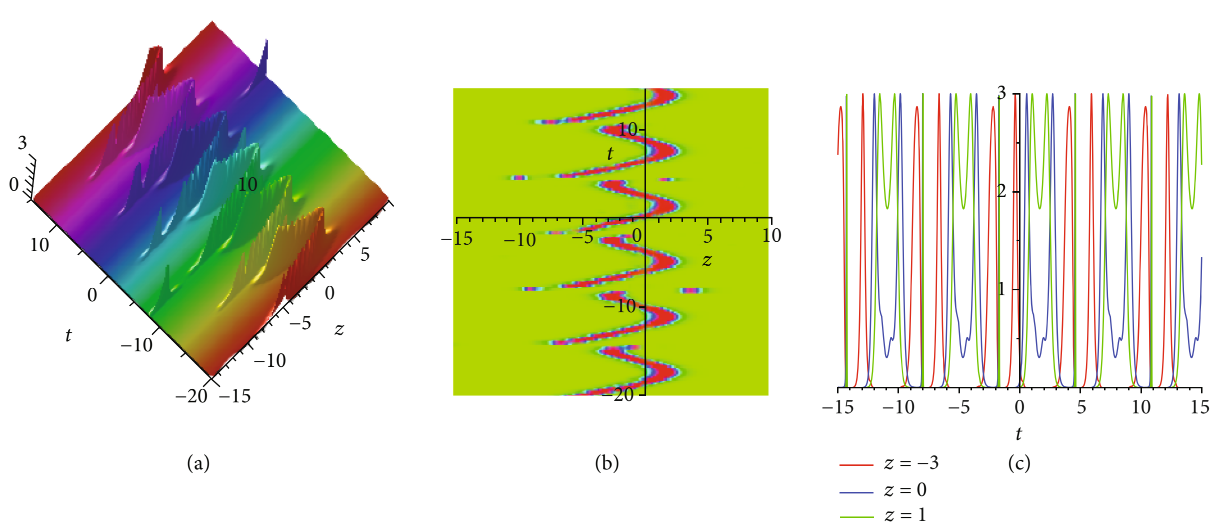


Figure 8: Plots of interaction lump with two solitons (49).

4.2. Interaction between Two Lump Solutions. Here, we offer interaction between two lump solutions containing combination of two functions for the $(3+1)$-dimensional variable-coefficient nonlinear wave equation through utilizing the bilinear method as the below frame:

$$g=\tau_1^4+\tau_2^2+\tau_3^2+\varepsilon_4(t), \tau_j=\alpha_j x+\beta_j y+\delta_j z+\varepsilon_j(t), j=1,2,3, \varepsilon_4(t)>0. \tag{50}$$

The values $\alpha_i, \beta_i, \delta_i, \varepsilon_i(t)(i=1,2,3)$ are real constants to be computed. By appending (50) into (22), we receive to a system containing 42 nonlinear equations. To solve the nonlinear system, the determined coefficients will be got as the below cases.

Type I

$$\begin{aligned}\varepsilon_1(t)&=\sqrt[4]{C_1-\varepsilon_4(t)}, \varepsilon_2(t)\\&=\int -2\frac{\beta_2\beta_3\phi_3(t)+\delta_2\delta_3\phi_4(t)}{\alpha_3}dt+C_2, \alpha_1=\alpha_2=0, \alpha_3=\alpha_3,\\ \varepsilon_3(t)&=\int -\frac{\alpha_3{}^2\phi_2(t)-\phi_3(t)\beta_2{}^2+\beta_3{}^2\phi_3(t)-\delta_2{}^2\phi_4(t)+\delta_3{}^2\phi_4(t)}{\alpha_3}dt\\&\quad+C_3, \beta_1=0, \beta_2=\beta_2,\\ \beta_3&=\beta_3, \delta_1=0, \phi_1(t)=-\frac{1}{3}\frac{\left(r_1{}^4+r_4\right)\left(\delta_2{}^2\phi_4(t)+\phi_3(t)\beta_2{}^2\right)}{\alpha_3{}^4}.\end{aligned} \tag{51}$$

The solutions are given as follows:

$$\begin{aligned}u_1&=12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_1), \mathbf{F}_1=(\varepsilon_1(t))^4+(\beta_2 y+\delta_2 z+\varepsilon_2(t))^2\\&\quad+(\alpha_3 x+\beta_3 y+\delta_3 z+\varepsilon_3(t))^2+\varepsilon_4(t).\end{aligned} \tag{52}$$

If $\tau_1^4+\tau_2^2+\tau_3^2+\varepsilon_4(t)\longrightarrow\infty$, the lump solutions $u\longrightarrow 0$ at any t. By selecting the parameters $\delta_2=2, \delta_3=3, \alpha_3=3, \beta_2=2, \beta_3=1, \phi_2(t)=t, \phi_3(t)=t^2, \phi_4(t)=1+t, x=1, z=1$, then plots of equation (52) are plotted in Figure 9. And also, by selecting the parameters $\delta_2=2, \delta_3=3, \alpha_3=3, \beta_2=2, \beta_3=1, \phi_2(t)=\cos(t), \phi_3(t)=\sin(t), \phi_4(t)=\cos(2t), x=1, z=1$, then plots of equation (52) are plotted in Figure 10. Moreover, by selecting the parameters $\delta_2=2, \delta_3=3, \alpha_3=3, \beta_2=2, \beta_3=1, \phi_2(t)=t^2\sin(2t), \phi_3(t)=t\sin(3t), \phi_4(t)=t\cos(3t), x=1, z=1$, then plots of equation (52) are plotted in Figure 11.

Type II

$$\begin{aligned}\varepsilon_1(t)&=\frac{\sqrt[4]{-\left(-C_1\alpha_2{}^3+\int F(t)\,dt\right)\alpha_2}}{\alpha_2}, \alpha_1=\beta_1=\delta_1=\phi_1(t)=0, \delta_3=\frac{\alpha_3\delta_2}{\alpha_2},\\ F(t)&=2\left(\phi_2(t)\alpha_2{}^2\alpha_3+\phi_3(t)\alpha_3\beta_2{}^2+\phi_4(t)\alpha_3\delta_2{}^2+\left(\frac{d}{dt}\varepsilon_3(t)\right)\alpha_2{}^2\right)(\varepsilon_3(t)\alpha_2-\varepsilon_2(t)\alpha_3)+\left(\frac{d}{dt}\varepsilon_4(t)\right)\alpha_2{}^3,\\ \varepsilon_2(t)&=\int -\frac{\left(\alpha_2{}^2+\alpha_3{}^2\right)\left(\delta_2{}^2\phi_4(t)+\phi_3(t)\beta_2{}^2+\phi_2(t)\alpha_2{}^2\right)+(d/dt\varepsilon_3(t))\alpha_2{}^2\alpha_3}{\alpha_2{}^3}dt+C_2.\end{aligned} \tag{53}$$

The solutions are given as follows:

$$\begin{aligned}u_2&=12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_2), \mathbf{F}_2=(\varepsilon_1(t))^4+(\alpha_2 x+\beta_2 y+\delta_2 z+\varepsilon_2(t))^2\\&\quad+\left(\alpha_3 x+\frac{\alpha_3\beta_2 y}{\alpha_2}+\frac{\alpha_3\delta_2 z}{\alpha_2}+\varepsilon_3(t)\right)^2+\varepsilon_4(t).\end{aligned} \tag{54}$$

Type III

$$\begin{aligned}\varepsilon_1(t)&=\int -2\frac{\phi_4(t)\delta_1(\beta_1\delta_3-\beta_3\delta_1)}{\alpha_3\beta_1}dt, \varepsilon_3(t)\\&=\int -\frac{\alpha_3{}^2\beta_1{}^2\phi_2(t)+\beta_1{}^2\delta_3{}^2\phi_4(t)-\beta_3{}^2\delta_1{}^2\phi_4(t)}{\alpha_3\beta_1{}^2}dt,\\ \varepsilon_4(t)&=-(\varepsilon_2(t))^2+C_1, \alpha_1=\alpha_2=\beta_2\\&=\delta_2=\phi_1(t)=0, \phi_3(t)=-\frac{\delta_1{}^2\phi_4(t)}{\beta_1{}^2}.\end{aligned} \tag{55}$$

The solutions are given as follows:

$$\begin{aligned}u_3&=12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_3), \mathbf{F}_3=(\beta_1 y+\delta_1 z+\varepsilon_1(t))^4+(\varepsilon_2(t))^2\\&\quad+(\alpha_3 x+\beta_3 y+\delta_3 z+\varepsilon_3(t))^2+\varepsilon_4(t).\end{aligned} \tag{56}$$

4.3. Interaction between Two Lumps-Soliton Solutions. Here, we offer interaction between two lumps-soliton solutions containing combination of two functions for the $(3+1)$-dimensional variable-coefficient nonlinear wave equation through utilizing the bilinear method as the below frame:

$$\begin{aligned}g&=\tau_1^4+\tau_2^2+\tau_3^2+\exp(\tau_4)+\varepsilon_5(t), \tau_j\\&=\alpha_j x+\beta_j y+\delta_j z+\varepsilon_j(t), j\\&=1,2,3,4, \varepsilon_4(t)>0.\end{aligned} \tag{57}$$

The values $\alpha_i, \beta_i, \delta_i, \varepsilon_i(t)(i=1,2,3,4)$ are real constants to be computed. By putting (57) into (22), we receive a

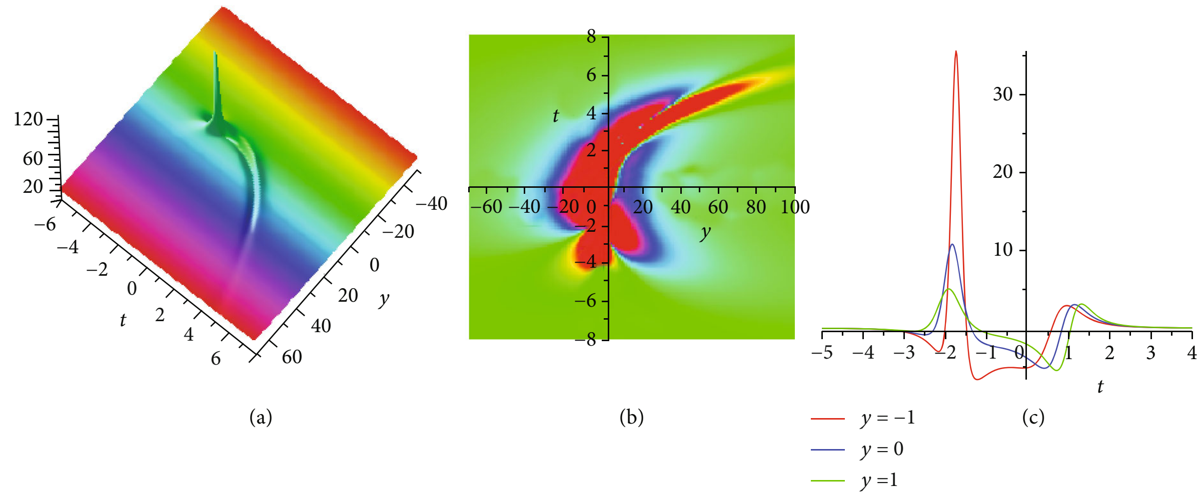


FIGURE 9: Plots of interaction with two lumps (52).

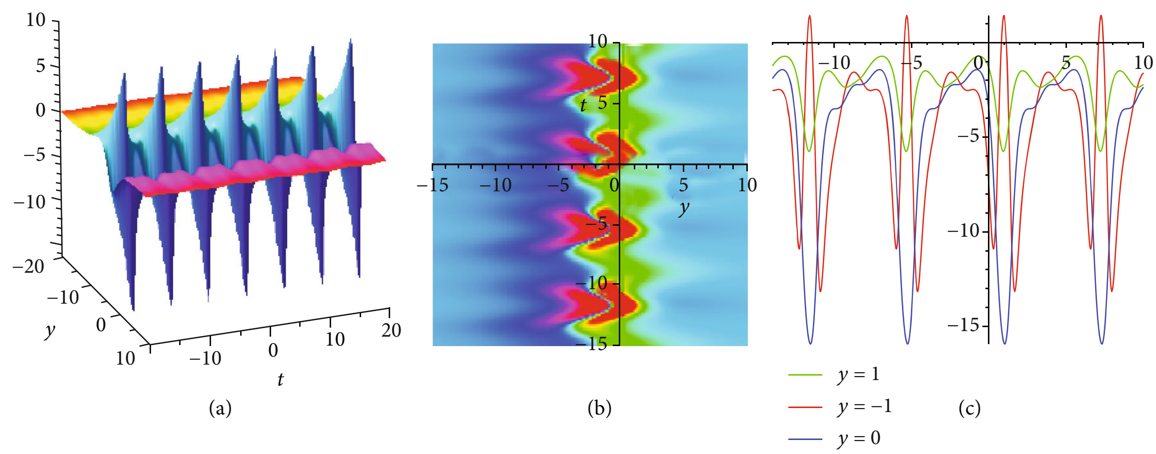


FIGURE 10: Plots of interaction with two lumps (52).

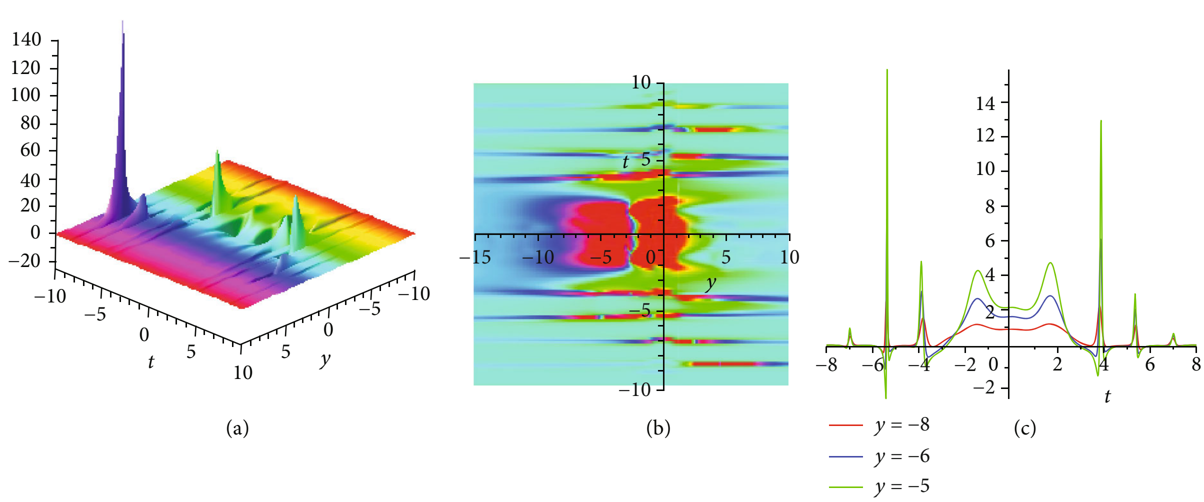


FIGURE 11: Plots of interaction with two lumps (52).

system containing 119 nonlinear equations. To solve the nonlinear system, the determined coefficients will be got as the below cases.

Type I

$$\begin{aligned}\varepsilon_1(t) &= \frac{\sqrt[4]{(A_1(t)+A_2(t))\alpha_2}}{\alpha_2}, A_1(t)\\ &= 2\,\alpha_3{}^2\beta_2{}^2\int \varepsilon_2(t)\phi_3(t)\,dt\\ &\quad - 2\,\alpha_3\int \varepsilon_3(t)\phi_2(t)\,dt\alpha_2{}^3 - 2\,\alpha_3\beta_2{}^2\int \varepsilon_3(t)\phi_3(t)dt\alpha_2,\\ A_2(t) &= 2\,\alpha_3{}^2\int \phi_2(t)\varepsilon_2(t)\,dt\alpha_2{}^2 + 2\,\alpha_3\int \varepsilon_2(t)\frac{d}{dt}\varepsilon_3(t)\,dt\alpha_2{}^2\\ &\quad - (\varepsilon_3(t))^2\alpha_2{}^3 - \varepsilon_5(t)\alpha_2{}^3 + C_1\alpha_2{}^3,\\ \varepsilon_2(t) &= \int -\frac{(\alpha_2{}^2+\alpha_3{}^2)(\phi_2(t)\alpha_2{}^2+\phi_3(t)\beta_2{}^2) + (d/dt\varepsilon_3(t))\alpha_2{}^2\alpha_3}{\alpha_2{}^3}dt\\ &\quad + C_2, \varepsilon_4(t) = 0,\\ &\alpha_1 = \alpha_4 = \beta_1 = \beta_4 = \delta_1 = \phi_1(t) = \phi_4(t) = 0, \beta_3 = \frac{\alpha_3\beta_2}{\alpha_2}, \delta_3 = \frac{\alpha_3\delta_2}{\alpha_2}.\end{aligned} \tag{58}$$

The solutions are given as follows:

$$\begin{aligned}u_1 &= 12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_1), \mathbf{F}_1 = (\varepsilon_1(t))^4 + (\alpha_2 x + \beta_2 y + \delta_2 z + \varepsilon_2(t))^2\\ &\quad + \left(\alpha_3 x + \frac{\alpha_3\beta_2 y}{\alpha_2} + \frac{\alpha_3\delta_2 z}{\alpha_2} + \varepsilon_3(t)\right)^2 + e^{\delta_4 z + \varepsilon_4(t)} + \varepsilon_5(t).\end{aligned} \tag{59}$$

If $\tau_1^4 + \tau_2^2 + \tau_3^2 + \exp(\tau_4) + \varepsilon_5(t) \longrightarrow \infty$, the lump solutions $u \longrightarrow 0$ at any t. By selecting the parameters $\delta_2 = 1$, $\delta_4 = 2, \alpha_2 = 1, \alpha_3 = 2, \beta_2 = 3, \phi_2(t) = \cos(t), \phi_3(t) = \sin(t)$, $\varepsilon_2(t) = \cos(2t), \varepsilon_3(t) = \sin(2t), x = 1, y = 1$, then plots of equation (59) are plotted in Figure 12. And also, by selecting the parameters $\delta_2 = 1, \delta_4 = 2, \alpha_2 = 1, \alpha_3 = 2, \beta_2 = 3, \phi_2(t) = \cos(1+t), \phi_3(t) = \sin(1+t), \varepsilon_2(t) = \cos(1+3t), \varepsilon_3(t) = \sin(1+3t), x = 1, y = 1$, then plots of equation (52) are plotted in Figure 13.

Type II

$$\begin{aligned}\varepsilon_1(t) &= \frac{\sqrt[4]{(A_1(t)+A_2(t)+A_3(t))\alpha_4{}^2\alpha_2{}^3}}{\alpha_4\alpha_2},\\ A_1(t) &= 2\,\alpha_3{}^2\beta_4{}^2\int \varepsilon_2(t)\phi_3(t)\,dt - 2\,\alpha_3\int \varepsilon_3(t)\phi_2(t)\,dt\alpha_4{}^2\alpha_2\\ &\quad - 2\,\alpha_3\beta_4{}^2\int \varepsilon_3(t)\phi_3(t)\,dt\alpha_2,\\ A_2(t) &= 2\,\alpha_3{}^2\int \phi_2(t)\varepsilon_2(t)\,dt\alpha_4{}^2 + 2\,\alpha_3{}^2\delta_4{}^2\int \phi_4(t)\varepsilon_2(t)\,dt\\ &\quad - 2\,\alpha_3\delta_4{}^2\int \phi_4(t)\varepsilon_3(t)\,dt\alpha_2,\\ A_3(t) &= 2\,\alpha_3\int \varepsilon_2(t)\frac{d}{dt}\varepsilon_3(t)\,dt\alpha_4{}^2 - (\varepsilon_3(t))^2\alpha_4{}^2\alpha_2 - \varepsilon_5(t)\alpha_4{}^2\alpha_2 + C_1\alpha_4{}^2\alpha_2,\\ \varepsilon_2(t) &= \int -\frac{(\alpha_2{}^2+\alpha_3{}^2)(\alpha_4{}^2\phi_2(t)+\beta_4{}^2\phi_3(t)+\delta_4{}^2\phi_4(t)) + (d/dt\varepsilon_3(t))\alpha_3\alpha_4{}^2}{\alpha_4{}^2\alpha_2}dt,\\ \varepsilon_4(t) &= \int -\frac{\alpha_4{}^2\phi_2(t)+\beta_4{}^2\phi_3(t)+\delta_4{}^2\phi_4(t)}{\alpha_4}dt,\\ &\alpha_1 = \beta_1 = \delta_1 = \phi_1(t) = 0, \beta_2 = \frac{\alpha_2\beta_4}{\alpha_4}, \beta_3 = \frac{\alpha_3\beta_4}{\alpha_4}, \delta_2 = \frac{\alpha_2\delta_4}{\alpha_4}, \delta_3 = \frac{\alpha_3\delta_4}{\alpha_4}.\end{aligned} \tag{60}$$

The solutions are given as follows:

$$u_2 = 12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_2), \tag{61}$$

$$\begin{aligned}\mathbf{F}_2 &= (\varepsilon_1(t))^4 + \left(\alpha_2 x + \frac{\alpha_2\beta_4 y}{\alpha_4} + \frac{\alpha_2\delta_4 z}{\alpha_4} + \varepsilon_2(t)\right)^2\\ &\quad + \left(\alpha_3 x + \frac{\alpha_3\beta_4 y}{\alpha_4} + \frac{\alpha_3\delta_4 z}{\alpha_4} + \varepsilon_3(t)\right)^2\\ &\quad + e^{\alpha_4 x + \beta_4 y + \delta_4 z + \varepsilon_4(t)} + \varepsilon_5(t).\end{aligned} \tag{62}$$

By selecting the parameters $\delta_4 = 2, \alpha_2 = 1, \alpha_3 = 2, \alpha_4 = 4$, $\beta_4 = 3, \phi_2(t) = \cos(t), \phi_3(t) = \sin(t), \phi_4(t) = \sin(2t), \varepsilon_2(t) = \cos(2t), \varepsilon_3(t) = \sin(2t), \varepsilon_4(t) = \sin(3t), C_1 = 1, C_2 = 2, x = 1, y = 1$, then plots of equation (61) are plotted in Figure 14. And also, by selecting the parameters $\delta_4 = 2, \alpha_2 = 1, \alpha_3 = 2, \alpha_4 = 4, \beta_4 = 3, \phi_2(t) = \cos(t), \phi_3(t) = \sin(t), \phi_4(t) = \sin(2t), \varepsilon_2(t) = t, \varepsilon_3(t) = t^2, \varepsilon_4(t) = t^3, C_1 = 1, C_2 = 2, x = 1, y = 1$, then plots of equation (54) are plotted in Figure 15.

4.4. Lump-Periodic Solutions. In this section, we would like to present the general solutions of the including combination of two functions for the $(3+1)$-dimensional variable-coefficient nonlinear wave equation through utilizing the bilinear method as the below frame:

$$\begin{aligned}g &= \tau_1^2 + \tau_2^2 + \cos(\tau_3) + \varepsilon_4(t), \tau_j\\ &= \alpha_j x + \beta_j y + \delta_j z + \varepsilon_j(t), j\\ &= 1, 2, 3, \varepsilon_4(t) > 0.\end{aligned} \tag{63}$$

The values $\alpha_i, \beta_i, \delta_i, \varepsilon_i(t)(i = 1, 2, 3)$ are real constants to be computed. By substituting (63) into (22), we obtain a system containing 24 nonlinear equations. To solve the nonlinear system, the determined coefficients will be got as the below cases.

Type I

$$\begin{aligned}\varepsilon_1(t) &= -\frac{\varepsilon_2(t)\alpha_2}{\alpha_1} - \frac{\alpha_2}{\alpha_1}\int \frac{\alpha_1{}^2\phi_2(t)}{\alpha_2} + \alpha_2\phi_2(t)\,dt + C_1, \varepsilon_3(t)\\ &= 0, \varepsilon_4(t) = -\frac{(\alpha_1{}^2+\alpha_2{}^2)\left(\left(\int \phi_2(t)\,dt\right)^2\alpha_2{}^2 + (\varepsilon_2(t))^2\right)}{\alpha_1{}^2}\\ &\quad - 2\frac{\varepsilon_2(t)\alpha_2}{\alpha_1{}^2}\left(\int \frac{\phi_2(t)(\alpha_1{}^2+\alpha_2{}^2)}{\alpha_2}dt\alpha_2 - C_1\alpha_1\right)\\ &\quad - 2\frac{\int -\phi_2(t)C_1\alpha_1\alpha_2{}^2\,dt}{\alpha_1{}^2} + C_2, \alpha_3\\ &= \phi_1(t) = \phi_3(t) = 0, \beta_2 = \frac{\alpha_2\beta_1}{\alpha_1}, \delta_2 = \frac{\alpha_2\delta_1}{\alpha_1}.\end{aligned} \tag{64}$$

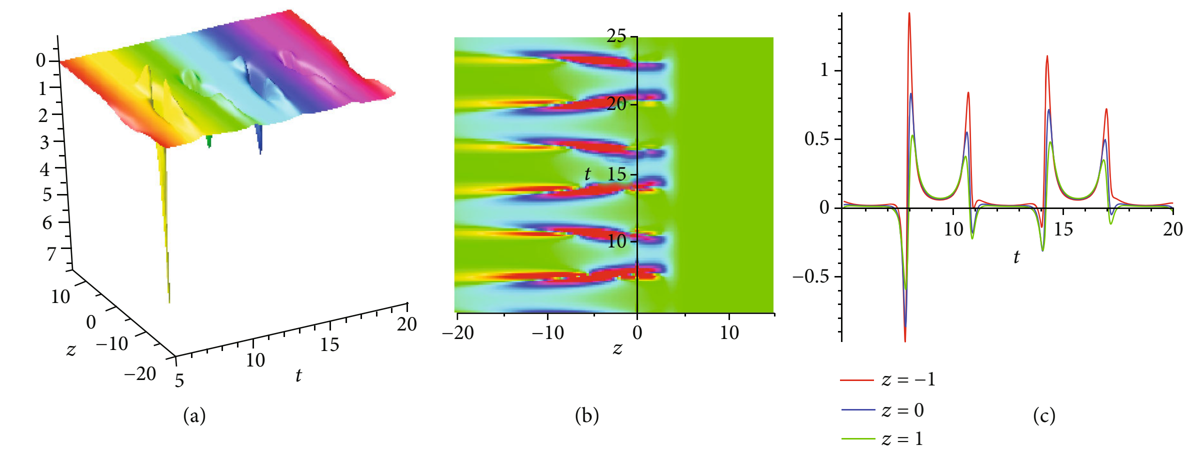


Figure 12: Plots of interaction with two lumps-soliton (59).

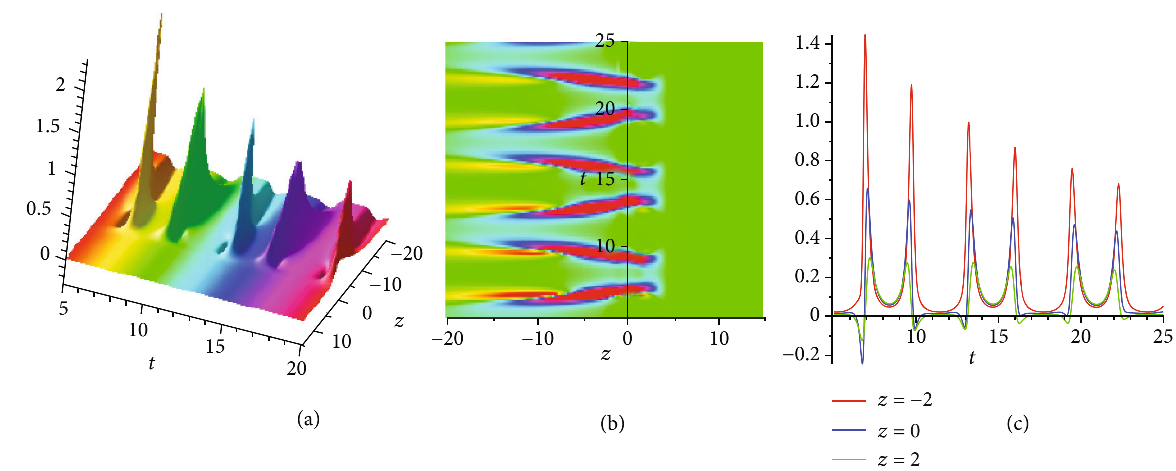


Figure 13: Plots of interaction with two lumps-soliton (59).

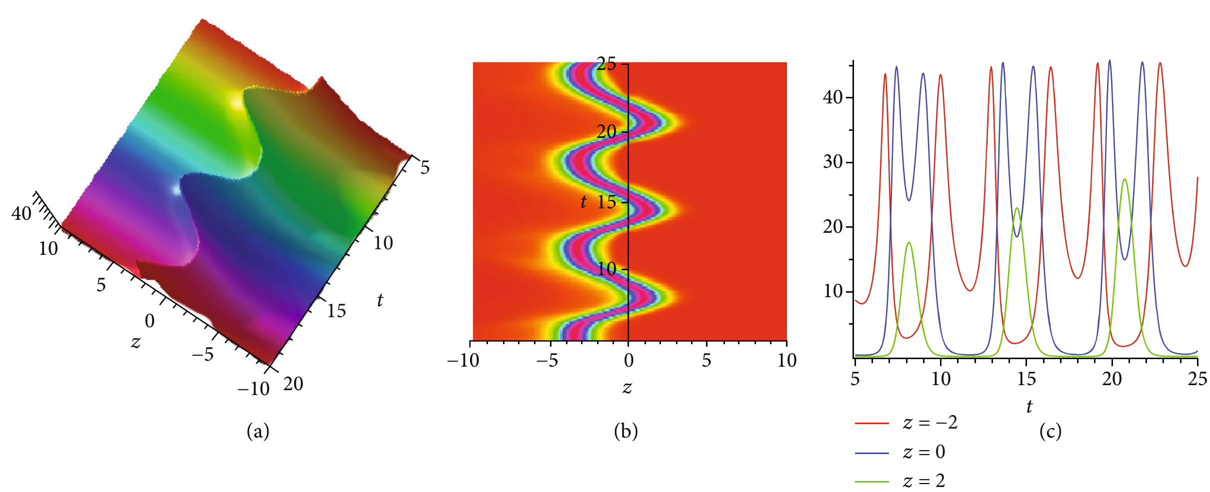


Figure 14: Plots of interaction with two lumps-soliton (61).

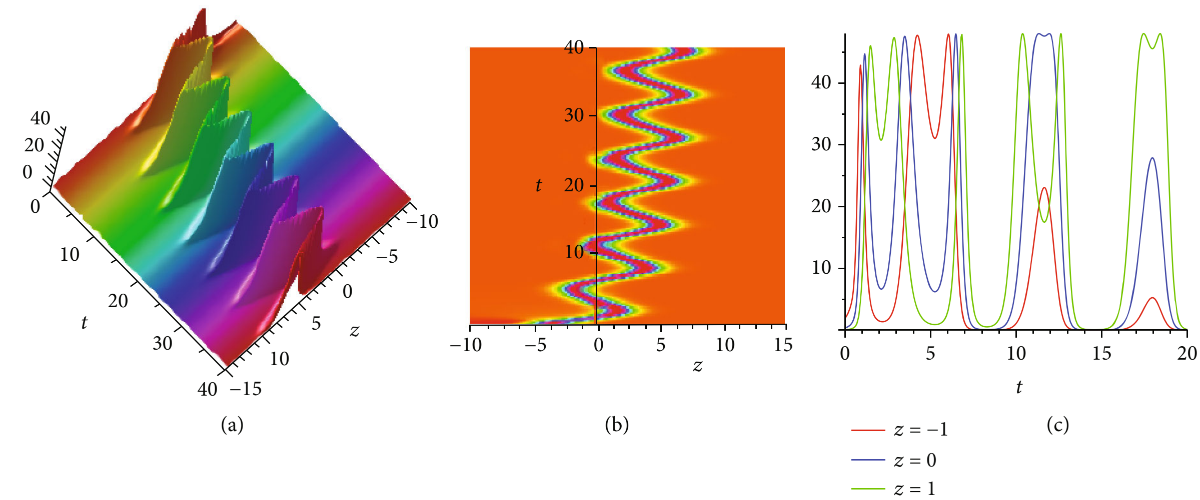


FIGURE 15: Plots of interaction with two lumps-soliton (61).

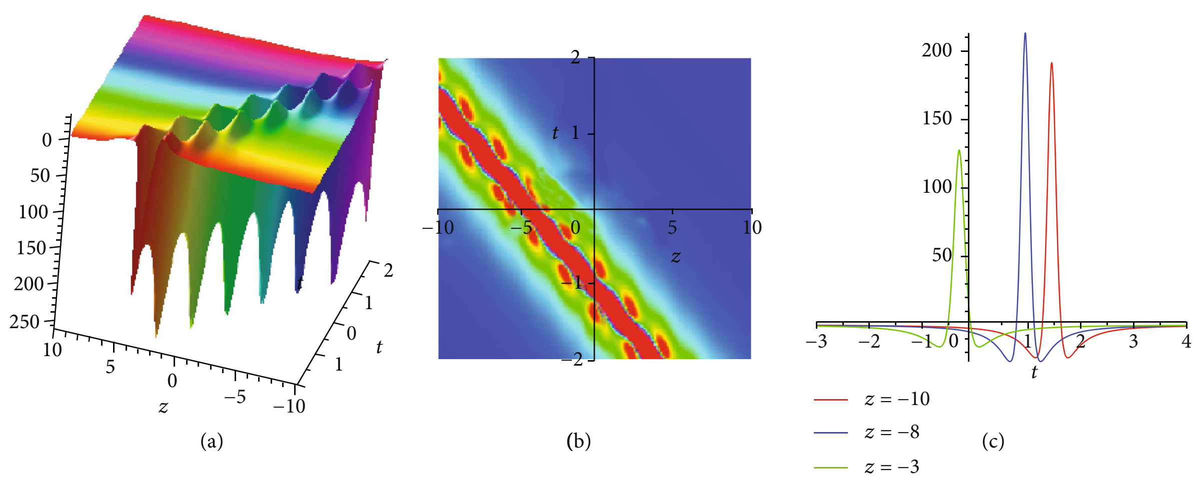


FIGURE 16: Plots of lump-periodic (65).

The solutions are given as follows:

$$u_1 = 12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_1), \mathbf{F}_1 = (\alpha_1 x + \beta_1 y + \delta_1 z + \varepsilon_1(t))^2 + \left(\alpha_2 x + \frac{\alpha_2\beta_1 y}{\alpha_1} + \frac{\alpha_2\delta_1 z}{\alpha_1} + \varepsilon_2(t)\right)^2 + \cos(\beta_3 y + \delta_3 z + \varepsilon_3(t)) + \varepsilon_4(t). \quad (65)$$

From above, we can discover that g is positive if $\varepsilon_4(t) > 0$, then in all space directions, u will be localized. Therefore, if $\tau_1^2 + \tau_2^2 + \cos(\tau_3) \longrightarrow \infty$, the lump solutions $u \longrightarrow 0$ at any t. By selecting the parameters $\delta_1 = 1, \delta_3 = 3, \alpha_1 = 2, \alpha_2 = 4, \beta_1 = 2, \beta_3 = 3, \phi_2(t) = -2, \varepsilon_2(t) = 3, C_1 = 1, C_2 = 2, x = 1, y = 1$, then plots of equation (65) are plotted in Figure 16.

Type II

$$\varepsilon_1(t) = \int -\frac{(\alpha_1^2 + \alpha_2^2)(\alpha_1^2\phi_2 + \delta_1^2\phi_4) + (d/dt\varepsilon_2(t))\alpha_1^2\alpha_2}{\alpha_1^3}dt + C_1, \varepsilon_3(t) = 0,$$
$$\varepsilon_4(t) = \int 2\frac{(\varepsilon_1(t)\alpha_2 - \varepsilon_2(t)\alpha_1)(\phi_2(t)\alpha_1^2\alpha_2 + \phi_4(t)\alpha_2\delta_1^2 + (d/dt\varepsilon_2(t))\alpha_1^2)}{\alpha_1^3}dt + C_2,$$
$$\alpha_3 = \delta_3 = \phi_1(t) = \phi_3(t) = 0, \beta_2 = \frac{\alpha_2\beta_1}{\alpha_1}, \delta_2 = \frac{\alpha_2\delta_1}{\alpha_1}. \quad (66)$$

The solutions are given as follows:

$$u_2 = 12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_2), \mathbf{F}_2 = (\alpha_1 x + \beta_1 y + \delta_1 z + \varepsilon_1(t))^2 + \left(\alpha_2 x + \frac{\alpha_2\beta_1 y}{\alpha_1} + \frac{\alpha_2\delta_1 z}{\alpha_1} + \varepsilon_2(t)\right)^2 + \cos(\beta_3 y + \varepsilon_3(t)) + \varepsilon_4(t). \quad (67)$$

Type III

$$\begin{aligned}\varepsilon_1(t) &= \int -2\frac{\delta_1\phi_4(t)(\beta_1\delta_2-\beta_2\delta_1)}{\alpha_2\beta_1}dt, \varepsilon_2(t)\\ &= \int -\frac{\alpha_2{}^2\beta_1{}^2\phi_2(t)+\beta_1{}^2\delta_2{}^2\phi_4(t)-\beta_2{}^2\delta_1{}^2\phi_4(t)}{\alpha_2\beta_1{}^2}dt,\\ \varepsilon_3(t) &= \int -2\frac{\delta_3\phi_4(t)(\beta_1\delta_2-\beta_2\delta_1)}{\alpha_2\beta_1}dt, \varepsilon_4(t)\\ &= 0, \beta_3 = \frac{\beta_1\delta_3}{\delta_1}, \phi_3(t) = -\frac{\delta_1{}^2\phi_4(t)}{\beta_1{}^2},\\ \alpha_1 &= \alpha_3 = \phi_1(t) = 0, \beta_2 = \frac{\alpha_2\beta_1}{\alpha_1}, \delta_2 = \frac{\alpha_2\delta_1}{\alpha_1}.\end{aligned} \tag{68}$$

The solutions are given as follows:

$$\begin{aligned}u_3 = 12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_3), \mathbf{F}_3 &= (\beta_1 y+\delta_1 z+\varepsilon_1(t))^2\\ &+ (\alpha_2 x+\beta_2 y+\delta_2 z+\varepsilon_2(t))^2\\ &+ \cos\left(\frac{\beta_1\delta_3 y}{\delta_1}+\delta_3 z+\varepsilon_3(t)\right)+\varepsilon_4(t).\end{aligned} \tag{69}$$

By selecting the parameters $\delta_1 = 1, \delta_2 = 2, \delta_3 = 3, \alpha_2 = 4, \beta_1 = 2, \beta_2 = 3, \phi_2(t) = \sin(t), \phi_4(t) = \cos(t), x = 1, y = 1$, then plots of equation (69) are plotted in Figure 17.

Type IV

$$\begin{aligned}\varepsilon_2(t) &= -\tan\left(\alpha_2\int\frac{\phi_2(t)}{\varepsilon_1(t)}dt + C_1\right)\varepsilon_1, \varepsilon_4(t)\\ &= \int 2\frac{\left((\varepsilon_1(t))^2+(\varepsilon_2(t))^2\right)(\alpha_2\phi_2(t)\varepsilon_2(t)-(d/dt\varepsilon_1(t))\varepsilon_1(t))}{(\varepsilon_1(t))^2}dt + C_2,\\ \varepsilon_3(t) &= 0, \alpha_1 = -\frac{\varepsilon_2(t)\alpha_2}{\varepsilon_1(t)}, \alpha_3 = \beta_1 = \beta_2 = \delta_1 = \delta_2\\ &= \phi_1(t) = 0, \phi_3(t) = -\frac{\delta_3{}^2\phi_4(t)}{\beta_3{}^2}.\end{aligned} \tag{70}$$

The solutions are given as follows:

$$\begin{aligned}u_4 = 12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_4), \mathbf{F}_4 &= \left(-\frac{\varepsilon_2(t)\alpha_2 x}{\varepsilon_1(t)}+\varepsilon_1(t)\right)^2\\ &+ (\alpha_2 x+\varepsilon_2(t))^2+\cos(\beta_3 y+\delta_3 z+\varepsilon_3(t))+\varepsilon_4(t).\end{aligned} \tag{71}$$

Type V

$$\begin{aligned}\varepsilon_2(t) &= -\tan\left(\frac{1}{\alpha_2\delta_3{}^2}\int\frac{\alpha_2{}^2\delta_3{}^2\phi_2(t)-\beta_3{}^2\delta_2{}^2\phi_3(t)}{\varepsilon_1(t)}dt\right)\varepsilon_1(t), \varepsilon_3(t) = \int 2\frac{\beta_3{}^2\delta_2\phi_3(t)}{\delta_3\alpha_2}dt,\\ \varepsilon_4(t) &= \int 2\frac{\left((\varepsilon_1(t))^2+(\varepsilon_2(t))^2\right)\left(\phi_2(t)\varepsilon_2(t)\alpha_2{}^2\delta_3{}^2-\phi_3(t)\varepsilon_2(t)\beta_3{}^2\delta_2{}^2-(d/dt\varepsilon_1(t))\alpha_2\delta_3{}^2\varepsilon_1(t)\right)}{\alpha_2\delta_3{}^2(\varepsilon_1(t))^2}dt,\\ \alpha_3 &= \beta_1 = \beta_2 = \phi_1(t) = 0, \alpha_1 = -\frac{\alpha_2\varepsilon_2(t)}{\varepsilon_1(t)}, \delta_1 = -\frac{\delta_2\varepsilon_2(t)}{\varepsilon_1(t)}, \phi_4(t) = -\frac{\beta_3{}^2\phi_3(t)}{\delta_3{}^2}.\end{aligned} \tag{72}$$

The solutions are given as follows:

$$u_5 = 12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_5), \mathbf{F}_5 = \left(-\frac{\alpha_2\varepsilon_2(t)x}{\varepsilon_1(t)}-\frac{\delta_2\varepsilon_2(t)z}{\varepsilon_1(t)}+\varepsilon_1(t)\right)^2+(\alpha_2 x+\delta_2 z+\varepsilon_2(t))^2+\cos(\beta_3 y+\delta_3 z+\varepsilon_3(t))+\varepsilon_4(t). \tag{73}$$

Type VI

$$\begin{aligned}&\varepsilon_3(t) = 0, \varepsilon_4(t) = -(\varepsilon_1(t))^2-(\varepsilon_2(t))^2,\\ &\alpha_3 = \beta_1 = \beta_2 = \phi_1(t) = \phi_3(t) = 0, \alpha_1 = -\frac{\alpha_2\varepsilon_2(t)}{\varepsilon_1(t)}, \delta_1 = -\frac{\delta_2\varepsilon_2(t)}{\varepsilon_1(t)}, \phi_4(t) = -\frac{\beta_3{}^2\phi_3(t)}{\delta_3{}^2},\\ &\phi_4(t) = \frac{\left(-\alpha_2\phi_2(t)\left((\varepsilon_1(t))^2+(\varepsilon_2(t))^2\right)+(d/dt\varepsilon_1(t))\varepsilon_1(t)\varepsilon_2(t)-(d/dt\varepsilon_2(t))(\varepsilon_1(t))^2\right)\alpha_2}{\left((\varepsilon_1(t))^2+(\varepsilon_2(t))^2\right)\delta_2{}^2}.\end{aligned} \tag{74}$$

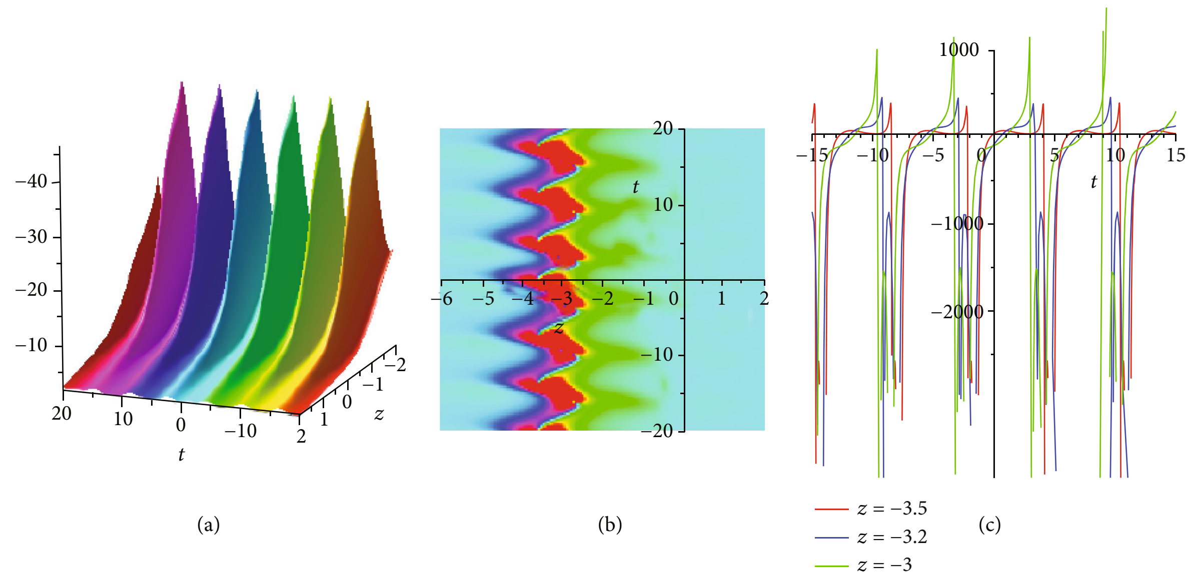


Figure 17: Plots of lump-periodic (69).

The solutions are given as follows:

$$u_6 = 12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_6), \mathbf{F}_6 = \left(-\frac{\alpha_2\varepsilon_2(t)x}{\varepsilon_1(t)} - \frac{\delta_2\varepsilon_2(t)z}{\varepsilon_1(t)} + \varepsilon_1(t)\right)^2 + (\alpha_2 x + \delta_2 z + \varepsilon_2(t))^2 + \cos(\beta_3 y) - (\varepsilon_1(t))^2 - (\varepsilon_2(t))^2. \quad (75)$$

Type VII

$$\varepsilon_1(t) = \int -\frac{\phi_2(t)(\alpha_1{}^2 + \alpha_2{}^2) + (d/dt\varepsilon_2(t))\alpha_2}{\alpha_1}dt + C_1, \varepsilon_3(t) = 0,$$

$$\varepsilon_4(t) = \int -2\,\frac{\alpha_2\phi_2(t)(\alpha_1\varepsilon_2(t) - \alpha_2\varepsilon_1(t)) + (d/dt\varepsilon_2(t))(\alpha_1\varepsilon_2(t) - \alpha_2\varepsilon_1(t))}{\alpha_1}dt + C_2,$$

$$\alpha_3 = \beta_1 = \beta_2 = \beta_3 = \phi_1(t) = \phi_4(t) = 0, \delta_2 = \frac{\alpha_2\delta_1}{\alpha_1}. \quad (76)$$

The solutions are given as follows:

$$u_7 = 12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_7), \mathbf{F}_7 = (\alpha_1 x + \delta_1 z + \varepsilon_1(t))^2 + \left(\alpha_2 x + \frac{\alpha_2\delta_1 z}{\alpha_1} + \varepsilon_2(t)\right)^2 + \cos(\delta_3 z + \varepsilon_3(t)) + \varepsilon_4(t). \quad (77)$$

By selecting the parameters $\delta_1 = 2, \alpha_1 = 1, \alpha_2 = 2, \beta_3 = 1$, $\phi_2(t) = t, \phi_4(t) = t^2, \varepsilon_1(t) = t, \varepsilon_2(t) = 1 + t, \varepsilon_3(t) = t^2, x = 1, y = 1$, then plots of equation (77) are plotted in Figure 18. And also, by selecting the parameters $\delta_1 = 2, \alpha_1 = 1, \alpha_2 = 2, \beta_3 = 1, \phi_2(t) = \sin(t), \phi_4(t) = \cos(t), \varepsilon_1(t) = t, \varepsilon_2(t) = \sin(t), \varepsilon_3(t) = \cos(t), x = 1, y = 1$, then plots of equation (77) are plotted in Figure 19. Moreover, by selecting the parameters $\delta_1 = 2, \alpha_1 = 1, \alpha_2 = 2, \beta_3 = 1, \phi_2(t) = 1, \phi_4(t) = 2, \varepsilon_1(t) = t, \varepsilon_2(t) = t, \varepsilon_3(t) = t^2, x = 1, y = 1$, then plots of equation (77) are plotted in Figure 20. And finally, by selecting the parameters $\delta_1 = 2, \alpha_1 = 1, \alpha_2 = 2, \beta_3 = 1, \phi_2(t) = \exp(t), \phi_4(t) = \exp(2t), \varepsilon_1(t) = \sinh(t), \varepsilon_2(t) = \cosh(t), \varepsilon_3(t) = t^2, x = 1, y = 1$, then plots of equation (77) are plotted in Figure 21.

4.5. Lump-Three Kink Solutions. Here, we present lump-periodic solutions containing combination of two functions for the $(3+1)$-dimensional variable-coefficient nonlinear wave equation through utilizing the bilinear method as the below frame:

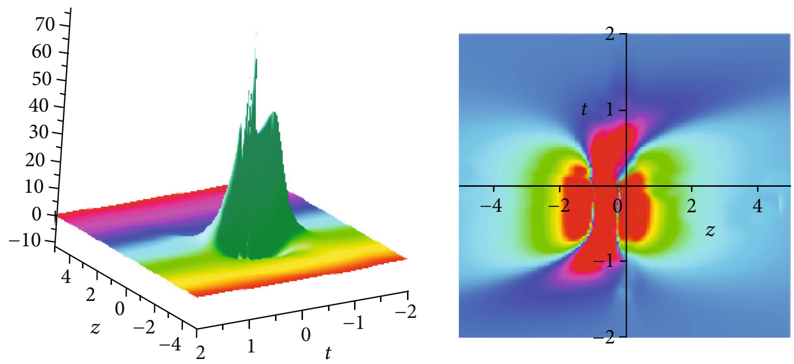
$$g = \tau_1^2 + \tau_2^2 + \exp(\tau_3) + \exp(\tau_4) + \exp(\tau_5) + \varepsilon_6(t), \tau_j = \alpha_j x + \beta_j y + \delta_j z + \varepsilon_j(t), j = 1, 2, 3, 4, 5, \varepsilon_6(t) > 0. \quad (78)$$

The values $\alpha_i, \beta_i, \delta_i, \varepsilon_i(t)(i = 1, \cdots, 5)$ are real constants to be computed. By appending (78) into (22), we receive a system containing 43 nonlinear equations. To solve the nonlinear system, the determined coefficients will be got as the below cases.

Type I

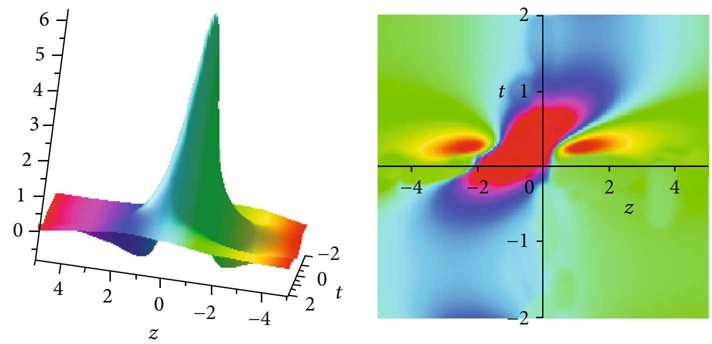
$$\varepsilon_1(t) = \int -2\,\frac{\beta_5\delta_1\phi_3(t)(\beta_2\delta_5 - \beta_5\delta_2)}{\delta_5{}^2\alpha_2}dt, \varepsilon_2(t) = \int -\frac{\alpha_2{}^2\delta_5{}^2\phi_2(t) + \beta_2{}^2\delta_5{}^2\phi_3(t) - \beta_5{}^2\delta_2{}^2\phi_3(t)}{\delta_5{}^2\alpha_2}dt,$$

$$\varepsilon_3(t) = \int -2\,\frac{\beta_5\delta_3\phi_3(t)(\beta_2\delta_5 - \beta_5\delta_2)}{\delta_5{}^2\alpha_2}dt, \varepsilon_4(t) = \int -2\,\frac{\beta_5\delta_4\phi_3(t)(\beta_2\delta_5 - \beta_5\delta_2)}{\delta_5{}^2\alpha_2}dt,$$



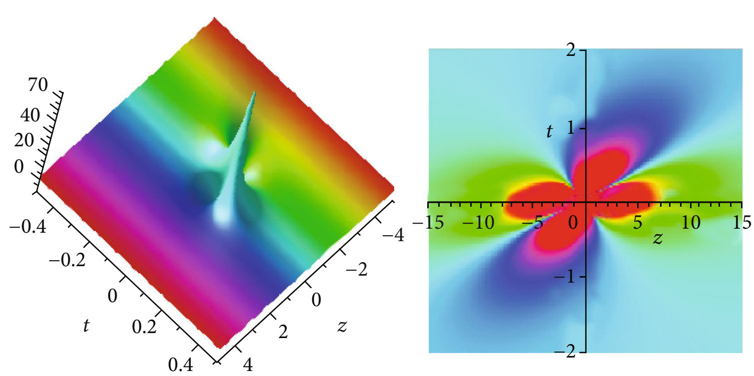
(a) (b)

Figure 18: Plots of lump-periodic (77).



(a) (b)

Figure 19: Plots of lump-periodic (77).



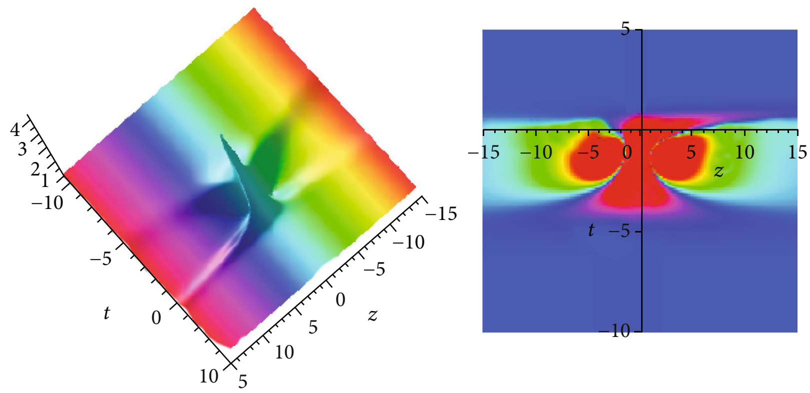
(a) (b)

Figure 20: Plots of lump-periodic (77).

$$\varepsilon_5(t) = \int -2\,\frac{\beta_5\phi_3(t)(\beta_2\delta_5 - \beta_5\delta_2)}{\alpha_2\delta_5}\,dt, \varepsilon_6(t) = 0, \alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = \phi_1(t) = 0, \beta_1 = \frac{\delta_1\beta_5}{\delta_5},$$
$$\beta_3 = \frac{\beta_5\delta_3}{\delta_5}, \beta_4 = \frac{\beta_5\delta_4}{\delta_5}, \phi_4(t) = -\frac{{\beta_5}^2\phi_3(t)}{{\delta_5}^2}. \tag{79}$$

The solutions are given as follows:

$$u_1 = 12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_1), \mathbf{F}_1 = \left(\frac{\delta_1\beta_5 y}{\delta_5} + \delta_1 z + \varepsilon_1(t)\right)^2 + (\alpha_2 x + \beta_2 y + \delta_2 z + \varepsilon_2(t))^2 + e^{\beta_5\delta_3 y/\delta_5 + \delta_3 z + \varepsilon_3(t)} + e^{\beta_5\delta_4 y/\delta_5 + \delta_4 z + \varepsilon_4(t)} + e^{\beta_5 y + \delta_5 z + \varepsilon_5(t)} + \varepsilon_6(t). \tag{80}$$



(a) (b)

Figure 21: Plots of lump-periodic (77).

Type II

$$\begin{aligned}
\varepsilon_1(t) &= \int -\frac{\left(\alpha_1^2+\alpha_2^2\right)\left(\alpha_4^2\phi_2(t)+\beta_4^2\phi_3(t)+\delta_4^2\phi_4(t)\right)+(d/dt\varepsilon_2(t))\alpha_2\alpha_4^2}{\alpha_1\alpha_4^2}dt,\\
\varepsilon_3(t) &= \int -\frac{\alpha_3\left(\alpha_4^2\phi_2(t)+\beta_4^2\phi_3(t)+\delta_4^2\phi_4(t)\right)}{\alpha_4^2}dt, \varepsilon_4(t) = \int -\frac{\alpha_4^2\phi_2(t)+\beta_4^2\phi_3(t)+\delta_4^2\phi_4(t)}{\alpha_4}dt,\\
\varepsilon_5(t) &= \int -\frac{\alpha_5\left(\alpha_4^2\phi_2(t)+\beta_4^2\phi_3(t)+\delta_4^2\phi_4(t)\right)}{\alpha_4^2}dt, \phi_1(t)=0, \beta_i=\frac{\alpha_i\beta_4}{\alpha_4}, \delta_i=\frac{\alpha_i\delta_4}{\alpha_4}, i=1,2,3,5,\\
\varepsilon_6(t) &= \int -2\frac{\alpha_2\left(\alpha_4^2\phi_2(t)+\beta_4^2\phi_3(t)+\delta_4^2\phi_4(t)\right)(\alpha_1\varepsilon_2(t)-\alpha_2\varepsilon_1(t))+(d/dt\varepsilon_2(t))\alpha_4^2(\alpha_1\varepsilon_2(t)-\alpha_2\varepsilon_1(t))}{\alpha_1\alpha_4^2}dt.
\end{aligned} \tag{81}$$

The solutions are given as follows:

$$\begin{aligned}
u_2 &= 12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_2), \mathbf{F}_2 = \left(\alpha_1 x+\frac{\alpha_1\beta_4 y}{\alpha_4}+\frac{\alpha_1\delta_4 z}{\alpha_4}+\varepsilon_1(t)\right)^2\\
&+\left(\alpha_2 x+\frac{\alpha_2\beta_4 y}{\alpha_4}+\frac{\alpha_2\delta_4 z}{\alpha_4}+\varepsilon_2(t)\right)^2\\
&+e^{\alpha_3 x+\alpha_3\beta_4 y/\alpha_4+\alpha_3\delta_4 z/\alpha_4+\varepsilon_3(t)}+e^{\alpha_4 x+\beta_4 y+\delta_4 z+\varepsilon_4(t)}\\
&+e^{\alpha_5 x+\alpha_5\beta_4 y/\alpha_4+\alpha_5\delta_4 z/\alpha_4+\varepsilon_5(t)}+\varepsilon_6(t).
\end{aligned} \tag{82}$$

Type III

$$\begin{aligned}
\varepsilon_1(t) &= \int -\frac{\phi_2(t)\left(\alpha_1^2+\alpha_2^2\right)}{\alpha_1}dt-\frac{\varepsilon_2(t)\alpha_2}{\alpha_1}, \varepsilon_3(t)\\
&= \int -\alpha_3\phi_2(t)\,dt, \varepsilon_4(t) = \int -\alpha_4\phi_2(t)dt,\\
\varepsilon_5(t) &= \int -\alpha_5\phi_2(t)dt, \varepsilon_6(t)\\
&= \int -2\frac{(\alpha_1\varepsilon_2(t)-\alpha_2\varepsilon_1(t))(\alpha_2\phi_2(t)+d/dt\varepsilon_2(t))}{\alpha_1}dt,
\end{aligned}$$

$$\begin{aligned}
\beta_2 &= \frac{\alpha_2\beta_1}{\alpha_1}, \beta_3=\frac{\beta_1\delta_3}{\delta_1}, \delta_2=\frac{\alpha_2\delta_1}{\alpha_1}, \delta_4=\frac{\beta_4\delta_1}{\beta_1}, \delta_5\\
&= \frac{\delta_1\beta_5}{\beta_1}, \phi_1(t)=0, \phi_4(t)=-\frac{\beta_1^2\phi_3(t)}{\delta_1^2}.
\end{aligned} \tag{83}$$

The solutions are given as follows:

$$\begin{aligned}
u_3 &= 12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_3), \mathbf{F}_3 = (\alpha_1 x+\beta_1 y+\delta_1 z+\varepsilon_1(t))^2\\
&+\left(\alpha_2 x+\frac{\alpha_2\beta_1 y}{\alpha_1}+\frac{\alpha_2\delta_1 z}{\alpha_1}+\varepsilon_2(t)\right)^2\\
&+e^{\alpha_3 x+\beta_1\delta_3 y/\delta_1+\delta_3 z+\varepsilon_3(t)}+e^{\alpha_4 x+\beta_4 y+\beta_4\delta_1 z/\beta_1+\varepsilon_4(t)}\\
&+e^{\alpha_5 x+\beta_5 y+\delta_1\beta_5 z/\beta_1+\varepsilon_5(t)}+\varepsilon_6(t).
\end{aligned} \tag{84}$$

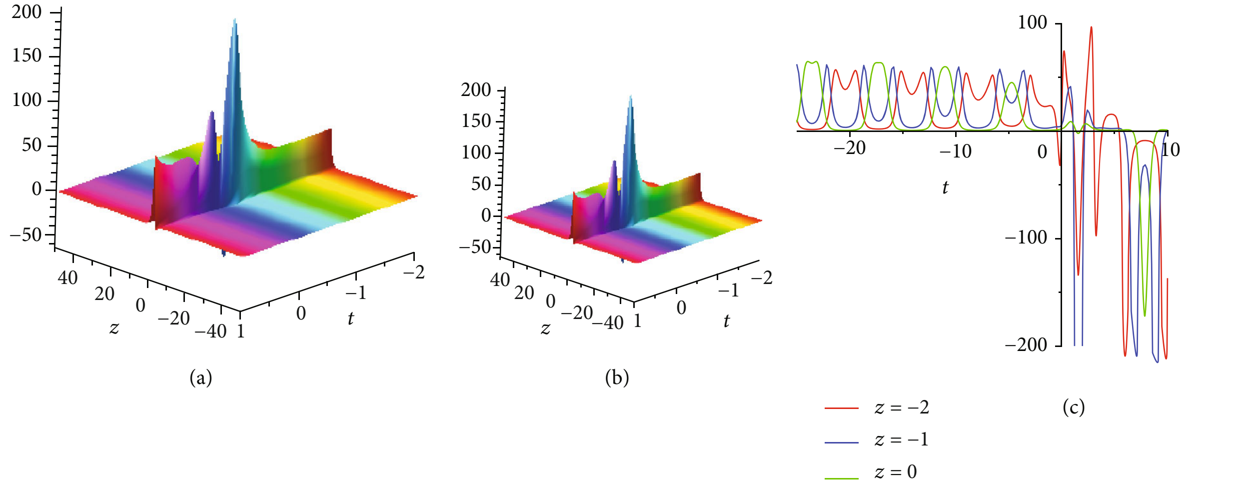


Figure 22: Plots of lump-three kink solutions (84).

From above, we can discover that g is positive if $\varepsilon_6(t) > 0$, then in all space directions u will be localized. Therefore, if $\tau_1^2 + \tau_2^2 + \exp(\tau_3) + \exp(\tau_4) + \exp(\tau_5) + \varepsilon_6(t) \longrightarrow \infty$, the lump-three kink solutions

$$u \Rightarrow \begin{cases} -12\alpha_3^2, & \alpha_3 > \alpha_4 > \alpha_5, \\ 0, & \alpha_4 > \alpha_3 > \alpha_3, \\ 0, & \alpha_5 > \alpha_3 > \alpha_4, \\ 0, & \alpha_4 > \alpha_5 > \alpha_3, \end{cases} \quad (85)$$

at any t. By selecting the parameters $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \alpha_4 = 4, \alpha_5 = 5, \beta_1 = 1, \beta_4 = 3, \beta_5 = 4, \delta_1 = 1, \delta_3 = 1.5, \phi_2(t) = \cos(t), \varepsilon_1(t) = \sin(2t), \varepsilon_2(t) = \cos(2t), x = 1, y = 1,$ then plots of equation (84) are plotted in Figure 22. And also, via selecting the parameters $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \alpha_4 = 4, \alpha_5 = 5, \beta_1 = 1, \beta_4 = 3, \beta_5 = 4, \delta_1 = 1, \delta_3 = 1.5, \phi_2(t) = 1 + t + t^2, \varepsilon_1(t) = 1/2(1+t)^2, \varepsilon_2(t) = 1/3(1+t)^3, x = 1, y = 1,$ then plots of equation (84) are plotted in Figure 23.

Type IV

$$\begin{aligned} \varepsilon_1(t) &= \int -\frac{(\alpha_1^2+\alpha_2^2)(\alpha_1^2\delta_4^2\phi_2(t) - \beta_4^2\delta_1^2\phi_3(t))}{\alpha_1^3\delta_4^2}dt - \frac{\varepsilon_2(t)\alpha_2}{\alpha_1}, \varepsilon_4(t) = \int 2\frac{\beta_4^2\phi_3(t)\delta_1}{\alpha_1\delta_4}dt, \\ \varepsilon_3(t) &= \int -\frac{\alpha_1^2\alpha_3\delta_4^2\phi_2(t) - \beta_4^2\delta_1\phi_3(t)(2\alpha_1\delta_3 - \alpha_3\delta_1)}{\alpha_1^2\delta_4^2}dt, \varepsilon_5(t) = \int -\frac{\alpha_1^2\alpha_5\delta_4^2\phi_2(t) - \beta_4^2\delta_1\phi_3(t)(2\alpha_1\delta_5 - \alpha_5\delta_1)}{\alpha_1^2\delta_4^2}dt, \\ \varepsilon_6(t) &= \int -2\frac{(\alpha_1\varepsilon_2(t) - \alpha_2\varepsilon_1(t))(\phi_2(t)\alpha_1^2\alpha_2\delta_4^2 - \phi_3(t)\alpha_2\beta_4^2\delta_1^2 + (d/dt\,\varepsilon_2(t))\alpha_1^2\delta_4^2)}{\alpha_1^3\delta_4^2}dt, \beta_3 = \frac{(\alpha_1\delta_3 - \alpha_3\delta_1)\beta_4}{\alpha_1\delta_4}, \\ \beta_5 &= \frac{\beta_4(\alpha_1\delta_5 - \alpha_5\delta_1)}{\alpha_1\delta_4}, \delta_2 = \frac{\alpha_2\delta_1}{\alpha_1}, \phi_4(t) = -\frac{\beta_4^2\phi_3(t)}{\delta_4^2}, \alpha_4 = \beta_1 = \beta_2 = \phi_1(t) = 0. \end{aligned} \quad (86)$$

The solutions are given as follows:

$$\begin{aligned} u_4 &= 12\frac{\partial^2}{\partial x^2}\ln(\mathbf{F}_4), \mathbf{F}_4 = (\alpha_1 x + \delta_1 z + \varepsilon_1(t))^2 \\ &+ \left(\alpha_2 x + \frac{\alpha_2\delta_1 z}{\alpha_1} + \varepsilon_2(t)\right)^2 + e^{\alpha_3 x + (\alpha_1\delta_3 - \alpha_3\delta_1)\beta_4 y/\delta_4\alpha_1 + \delta_3 z + \varepsilon_3(t)} \\ &+ e^{\beta_4 y + \delta_4 z + \varepsilon_4(t)} + e^{\alpha_5 x + \beta_4(\alpha_1\delta_5 - \alpha_5\delta_1)y/\delta_4\alpha_1 + \delta_5 z + \varepsilon_5(t)} + \varepsilon_6(t). \end{aligned} \quad (87)$$

By selecting the parameters $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \alpha_5 = 5, \beta_4 = 3, \delta_1 = 1, \delta_3 = 1.5, \delta_4 = 2, \delta_5 = 3, \phi_2(t) = t, \phi_3(t) = t^2, \varepsilon_2(t) = t^4, x = 1, y = 1,$ then plots of equation (87) are plotted in Figure 24. And also, via selecting the parameters $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \alpha_5 = 5, \beta_4 = 3, \delta_1 = 1, \delta_3 = 1.5, \delta_4 = 2, \delta_5 = 3, \phi_2(t) = 1, \phi_3(t) = 2, \varepsilon_2(t) = 3, x = 1, y = 1,$ then plots of equation (87) are plotted in Figure 25. Moreover, via selecting the parameters $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \alpha_5 = 5, \beta_4 = 3, \delta_1 = 1, \delta_3$

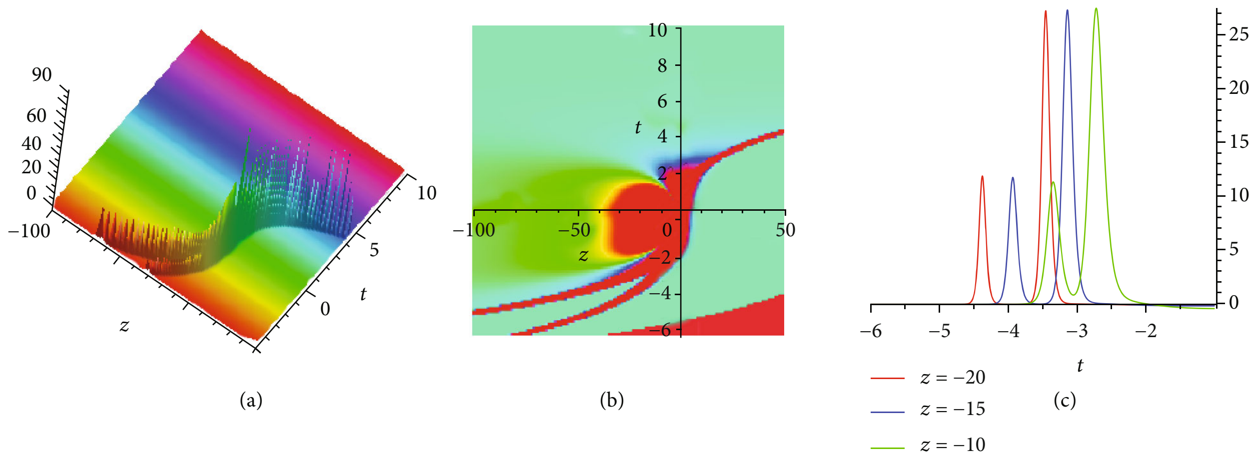


FIGURE 23: Plots of lump-three kink solutions (84).

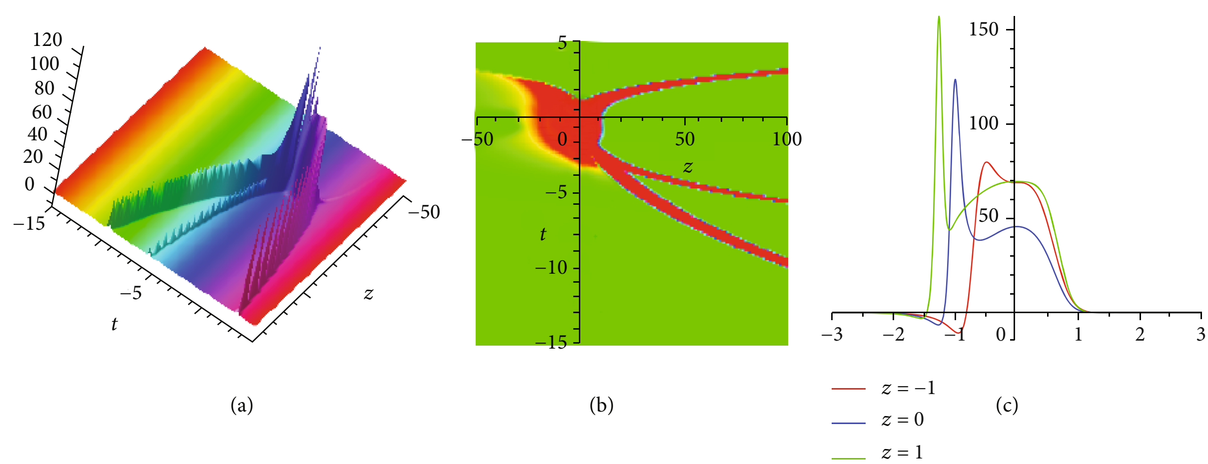


FIGURE 24: Plots of lump-three kink solutions (87).

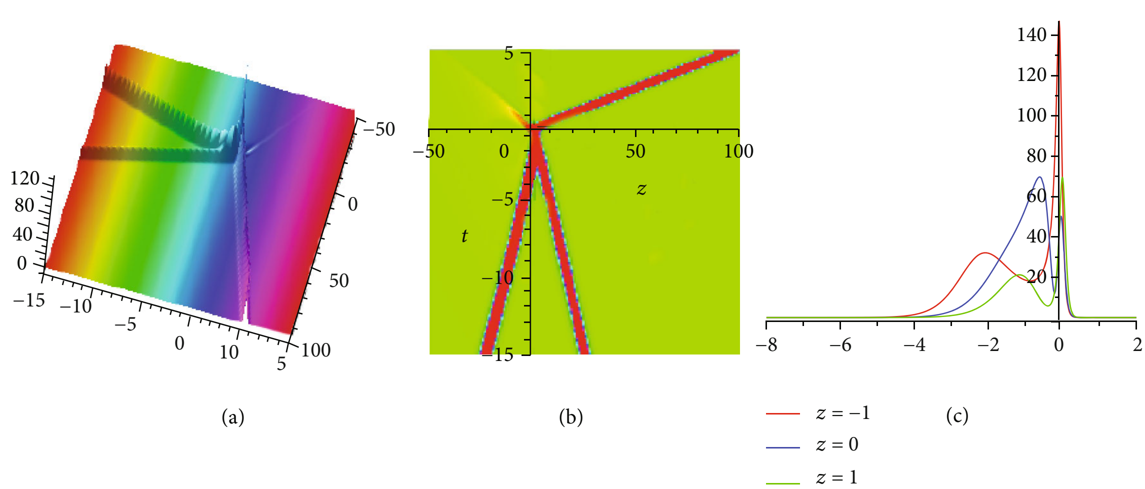


FIGURE 25: Plots of lump-three kink solutions (87).

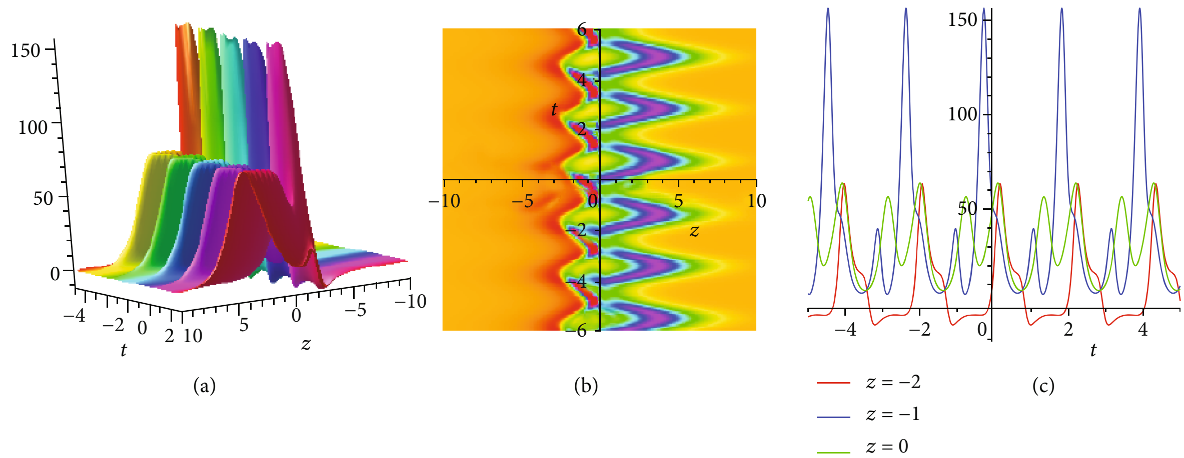


FIGURE 26: Plots of lump-three kink solutions (87).

$= 1.5, \delta_4 = 2, \delta_5 = 3, \phi_2(t) = \sin(3t), \phi_3(t) = \cos(3t), \varepsilon_2(t) = \sin(t) + \cos(t), x = 1, y = 1$, then plots of equation (87) are plotted in Figure 26.

5. Conclusion

In this paper, under the multidimensional binary Bell polynomials, we derive that the lump-soliton and its interaction solutions of the $(3+1)$-dimensional variable-coefficient nonlinear wave equation in liquid with gas bubbles has been successfully obtained. And then, ten classes for interaction between a lump-two kink solitons, three classes for interaction between two lumps, two classes for interaction between two lumps-soliton, seven classes for lump-periodic, and four classes for lump-three kink solutions were successfully constructed by employing the bilinear scheme. With the combination of the binary Bell polynomials and Hirota's bilinear method, some concrete solutions are expressed in terms of determined functions. We illustrate the regularity of solutions with some parameter constrains on constant and rational and periodic backgrounds. We also consider the classical dynamical behaviors and the elasticity of the collisions of soliton solutions and their interactions when plotting them. Finally, we conclude two dynamical behaviors of general lump-periodic and multikink soliton solutions.

Data Availability

The datasets supporting the conclusions of this article are included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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